ECON2001 - Economics 2A Micro

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1 Preliminaries

Unless stated otherwise, I will assume throughout my lectures that:

- There is a consumer who is endowed with an income equal to $m \geq 0$
- There are two perfectly divisible items $i = 1, 2$
- A typical quantity of item 1 is denoted by $x_1 \geq 0$ (horizontal line)
- A typical quantity of item 2 is denoted by $x_2 \geq 0$ (vertical line)
- A typical discrete change in the quantity of item $i = 1, 2$ is denoted by Δx_i
- A typical infinitesimal (infinitely small) change in the quantity of item $i = 1, 2$ is denoted by dx_i
- A typical consumption bundle is denoted by the pair (x_1, x_2)
- A typical price for one unit of consumption of item $i = 1, 2$ is denoted by $p_i > 0$

2 Budget Constraint

Definition 1 *The **set of affordable bundles** for a consumer who has an income m and who faces a list of prices (p_1, p_2) consists of consumption bundles (x_1, x_2) such that it holds that:*

$$p_1x_1 + p_2x_2 \leq m$$

Definition 2 *The **budget line** of a consumer who has an income m and who faces a list of prices (p_1, p_2) is the set of affordable bundles that cost exactly m, that is,*

$$p_1x_1 + p_2x_2 = m \qquad (1)$$

Remark 1 *The **budget line** in (1) can be rearranged to give us the formula of a straight line*

$$x_2 = \underbrace{\frac{m}{p_2}}_{\text{Vertical Intercept}} - \underbrace{\frac{p_1}{p_2}}_{\text{Slope}} x_1 \qquad (2)$$

***Remark 2** If the consumer spends all his income on item $i = 1, 2$, then the consumer can buy $\frac{m}{p_i}$ units of item i*

***Remark 3** The slope of the budget line $\left(-\frac{p_1}{p_2}\right)$ measures the rate at which the mkt is willing to "substitute" item 1 for item 2. It can also be interpreted as the **opportunity cost of consuming item 1**. The reason for this is that for any affordable consumption bundle (x_1, x_2) such that*

$$p_1 x_1 + p_2 x_2 = m, \tag{3}$$

and for any other affordable consumption bundle $(x_1 + \Delta x_1, x_2 + \Delta x_2)$ such that

$$p_1 (x_1 + \Delta x_1) + p_2 (x_2 + \Delta x_2) = m, \tag{4}$$

the total value of the change in consumption is

$$p_1 \Delta x_1 + p_2 \Delta x_2 = 0.$$

Solving for $\frac{\Delta x_2}{\Delta x_1}$, the mkt rate at which item 2 (item 1) can be substituted for item 1 (item 2) while still satisfying the budget line, gives

$$\frac{\Delta x_2}{\Delta x_1} = -\frac{p_1}{p_2}$$

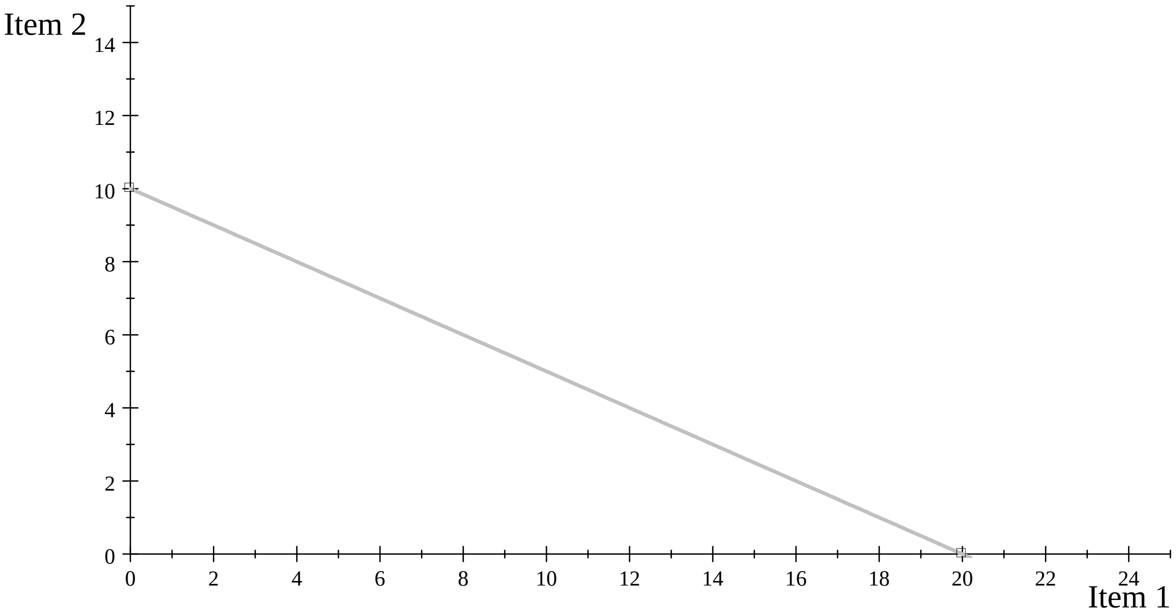
Example 1 *Suppose that Alice has an income of $m = £100$. The price of one unit of item 1 is $p_1 = £5$ and the price of a unit of item 2 is $p_2 = £10$. Then, the* ***set of affordable bundles*** *is*

$$5x_1 + 10x_2 \leq 100.$$

The ***budget line*** *is*

$$x_2 = 10 - 0.5x_1.$$

Graphically,



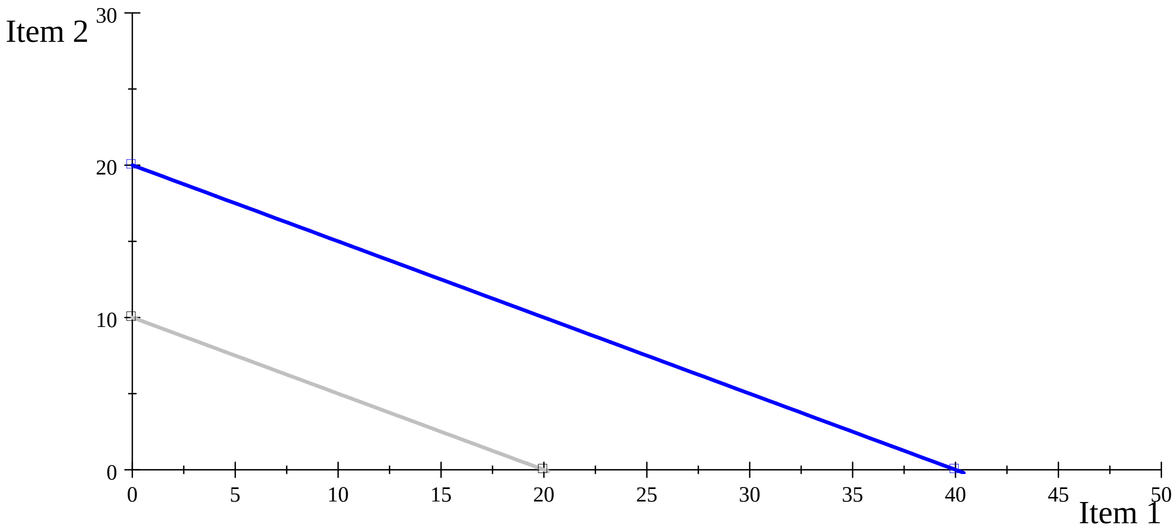
Example 2 *Initially, suppose that Alice has an income of* $m = £100$*, that the price of one unit of item* 1 *is* $p_1 = £5$ *and that the price of a unit of item 2 is* $p_2 = £10$*. Now, suppose that only her income increases to* $m' = £200$*. Then, the* ***new set of affordable bundles*** *is*

$$5x_1 + 10x_2 \leq 200.$$

The ***new budget line*** *(blue line) is*

$$x_2 = 20 - 0.5x_1.$$

Graphically,



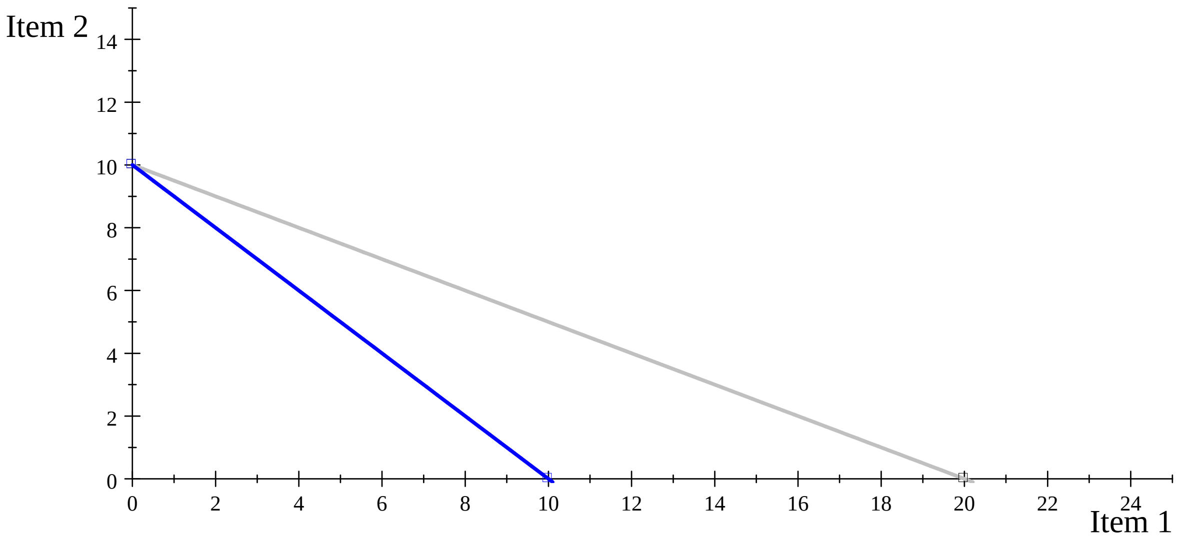
Example 3 *Initially, suppose that Alice has an income of* $m = £100$*, that the price of one unit of item* 1 *is* $p_1 = £5$ *and that the price of a unit of item 2 is* $p_2 = £10$*. Now, suppose that only the price of item 1 increases to* $p_1' = £10$*. Then, the* ***new set of affordable bundles*** *is*

$$10x_1 + 10x_2 \leq 100.$$

The ***new budget line*** *(blue line) is*

$$x_2 = 10 - x_1.$$

Graphically,



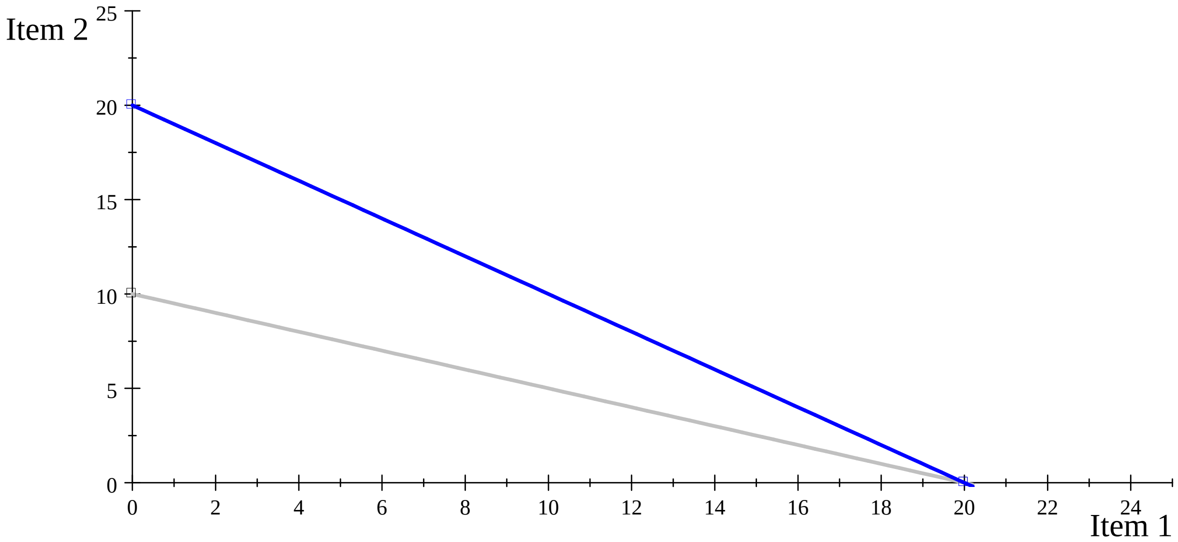
Example 4 *Initially, suppose that Alice has an income of* $m = \pounds 100$*, that the price of one unit of item* 1 *is* $p_1 = \pounds 5$ *and that the price of a unit of item 2 is* $p_2 = \pounds 10$*. Now, suppose that only the price of item 2 decreases to* $p_2' = \pounds 5$*. Then, the* ***new set of affordable bundles*** *is*

$$5x_1 + 5x_2 \leq 100.$$

The ***new budget line*** *(blue line) is*

$$x_2 = 20 - x_1.$$

Graphically,



***Definition 3** A **quantity tax** t is a tax that is defined as a fixed amount for each unit of a good or service consumed. Suppose that a quantity tax t is set on item 1. The set of affordable bundles for a consumer who has an income m and who faces a list of prices (p_1, p_2) consists of consumption bundles (x_1, x_2) such that it holds that:*

$$(p_1 + t)\, x_1 + p_2 x_2 \leq m$$

***Definition 4** A **value tax** τ is a tax on the value of the good or service. Suppose that a value tax rate τ is set on item 1. The set of affordable bundles for a consumer who has an income m and who faces a list of prices (p_1, p_2) consists of consumption bundles (x_1, x_2) such that it holds that:*

$$(1 + \tau)\, p_1 x_1 + p_2 x_2 \leq m$$

***Definition 5** A **quantity subsidy** s is a subsidy that is defined as fix amount that the consumer receives from the government for each unit of the good he purchases. Suppose that a quantity subsidy s is set on item 1. The set of affordable bundles for a consumer who has an income m and who faces a list of prices (p_1, p_2) consists of consumption bundles (x_1, x_2) such that it holds that:*

$$(p_1 - s)\, x_1 + p_2 x_2 \leq m$$

***Definition 6** A **value subsidy** σ is a subsidy on the value of the good or service. Suppose that a value subsidy rate σ is set on item 1. The set of affordable bundles for a consumer who has an income m and who faces a list of prices (p_1, p_2) consists of consumption bundles (x_1, x_2) such that it holds that:*

$$(1 - \sigma)\, p_1 x_1 + p_2 x_2 \leq m$$

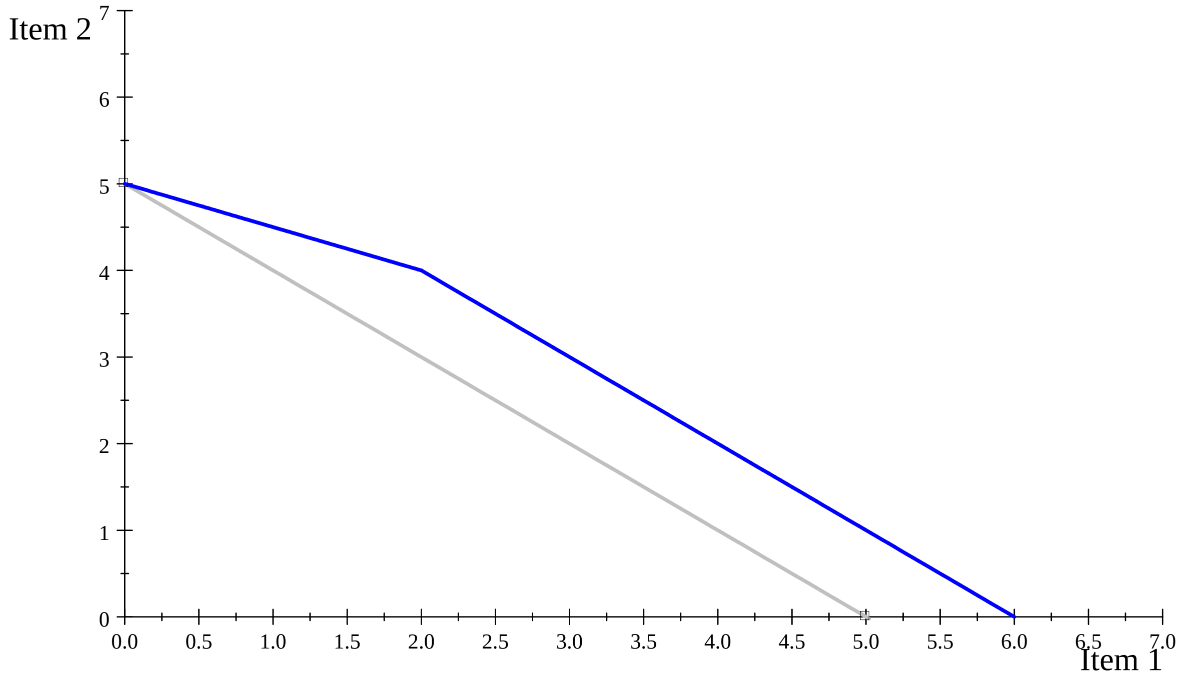
Definition 7 *A lump-sum **tax** T is a tax that is defined as a fixed amount that the government takes away from the consumer, regardless of the consumer's behavior. Suppose that a lump-sum tax T is set on consumer's income. The set of affordable bundles for a consumer who has an income m and who faces a list of prices (p_1, p_2) consists of consumption bundles (x_1, x_2) such that it holds that:*

$$p_1 x_1 + p_2 x_2 \leq m - T$$

Definition 8 *A **rationing** is the artificial restriction of raw materials, goods or services. Suppose that item 1 is rationed so that no more that $\bar{x}_1$ units can be consumed by a consumer. The set of affordable bundles for a consumer who has an income m and who faces a list of prices (p_1, p_2) consists of consumption bundles (x_1, x_2) such that it holds that:*

$$p_1 x_1 + p_2 x_2 \leq m \text{ and } x_1 \leq \bar{x}_1$$

Example 5 *Initially, suppose that Alice has an income of $m = £5$, that the price of one unit of item 1 is $p_1 = £1$ and that the price of a unit of item 2 is $p_2 = £1$. Now, suppose that Alice gets one (and only one) coupon that allows her to buy two units of item 1. Graphically (blue line is the new budget line),*



3 Preferences

Assumption 1 *The consumer can rank consumption bundles. We denote the* ***consumer's preference*** *(ranking) by the symbol $\succsim$, where $\succ$ is the strict preference part of $\succsim$ and $\sim$ is the indifference preference part of $\succsim$*

For every two consumption bundles (x_1, x_2) and (y_1, y_2):

- $(x_1, x_2) \succsim (y_1, y_2)$ means that (x_1, x_2) is **at least as good as** (y_1, y_2)
- $(x_1, x_2) \succ (y_1, y_2)$ means that (x_1, x_2) is **strictly better than** (y_1, y_2)
- $(x_1, x_2) \sim (y_1, y_2)$ means that (x_1, x_2) is **as good as** (y_1, y_2)

Definition 9 *A consumer's preference $\succsim$ over consumption bundles is a* ***well-behaved preference*** *if it satisfies the following properties for every three bundles (x_1, x_2), (y_1, y_2) and (z_1, z_2):*

Completeness*: Either $(x_1, x_2) \succ (y_1, y_2)$, or $(y_1, y_2) \succ (x_1, x_2)$, or $(x_1, x_2) \sim (y_1, y_2)$*

Reflexivity*: $(x_1, x_2) \succsim (x_1, x_2)$*

Transitivity*:*

- $(x_1, x_2) \succ (y_1, y_2) \succ (z_1, z_2) \implies (x_1, x_2) \succ (z_1, z_2)$
- $(x_1, x_2) \sim (y_1, y_2) \sim (z_1, z_2) \implies (x_1, x_2) \sim (z_1, z_2)$
- $(x_1, x_2) \succ (y_1, y_2) \sim (z_1, z_2) \implies (x_1, x_2) \succ (z_1, z_2)$
- $(x_1, x_2) \sim (y_1, y_2) \succ (z_1, z_2) \implies (x_1, x_2) \succ (z_1, z_2)$

Monotonicity *(more is strictly better!):*
$(x_1, x_2) \neq (y_1, y_2)$ and $(x_1, x_2) \geq (y_1, y_2) \implies (x_1, x_2) \succ (y_1, y_2)$

Convexity *(the weighted-average bundle is at least as good as the extreme bundles):*
For every number t such that $0 \leq t \leq 1$,

$$(y_1, y_2) \sim (x_1, x_2) \quad \implies (tx_1 + (1-t)\, y_1, tx_2 + (1-t)\, y_2) \succsim (x_1, x_2)$$

Definition 10 *Given a consumption bundle* (x_1, x_2) *and a consumer's preference* $\succsim$ *over consumption bundles, the* ***indifference curve at*** (x_1, x_2), *denoted by* $IC(x_1, x_2)$, *consists of bundles that are as good as the bundle* (x_1, x_2), *that is,*

$$IC(x_1, x_2) = \{(y_1, y_2) \mid (x_1, x_2) \sim (y_1, y_2)\}$$

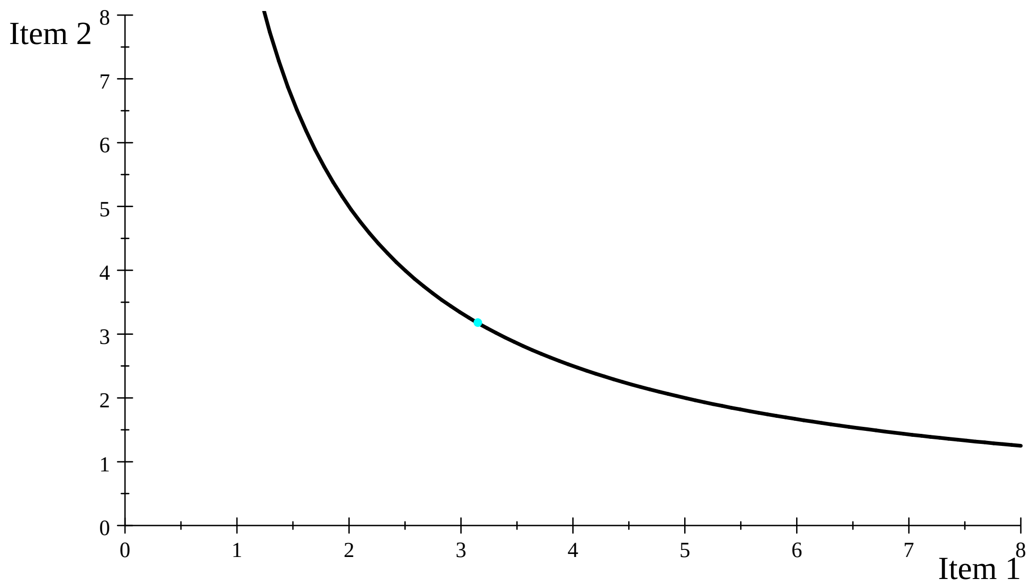
Definition 11 *Given a consumption bundle* (x_1, x_2) *and a consumer's preference* $\succsim$ *over consumption bundles, the* ***upper contour set at*** (x_1, x_2), *denoted by* $UC(x_1, x_2)$, *consists of bundles that are at least as good as the bundle* (x_1, x_2), *that is,*

$$UC(x_1, x_2) = \{(y_1, y_2) \mid (y_1, y_2) \succsim (x_1, x_2)\}$$

Definition 12 *Given a consumption bundle* (x_1, x_2) *and a consumer's preference* $\succsim$ *over consumption bundles, the* ***lower contour set at*** (x_1, x_2), *denoted by* $LC(x_1, x_2)$, *consists of bundles for which the bundle* (x_1, x_2) *is at least as good as, that is,*

$$LC(x_1, x_2) = \{(y_1, y_2) \mid (x_1, x_2) \succsim (y_1, y_2)\}$$

Remark 4 *If a consumer has a monotonic preference, then the only graph compatible with his preference is a downward-sloping (thin) indifference curve*

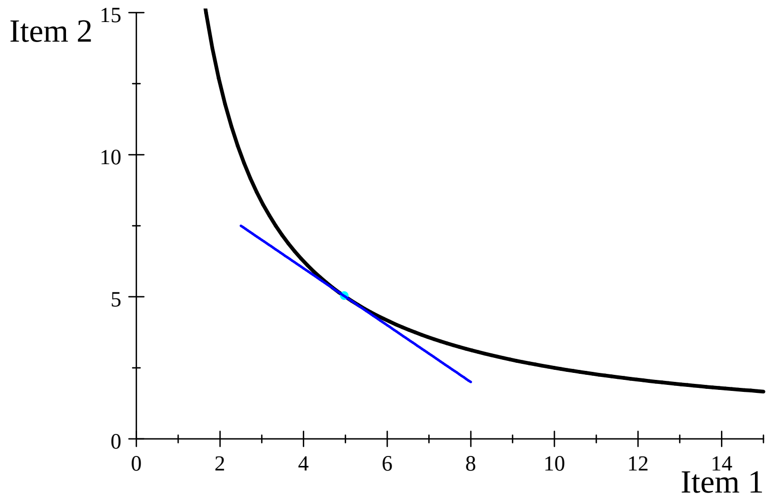


Remark 5 *If a consumer has a monotonic and transitive preference, then his indifference curves cannot cross.*

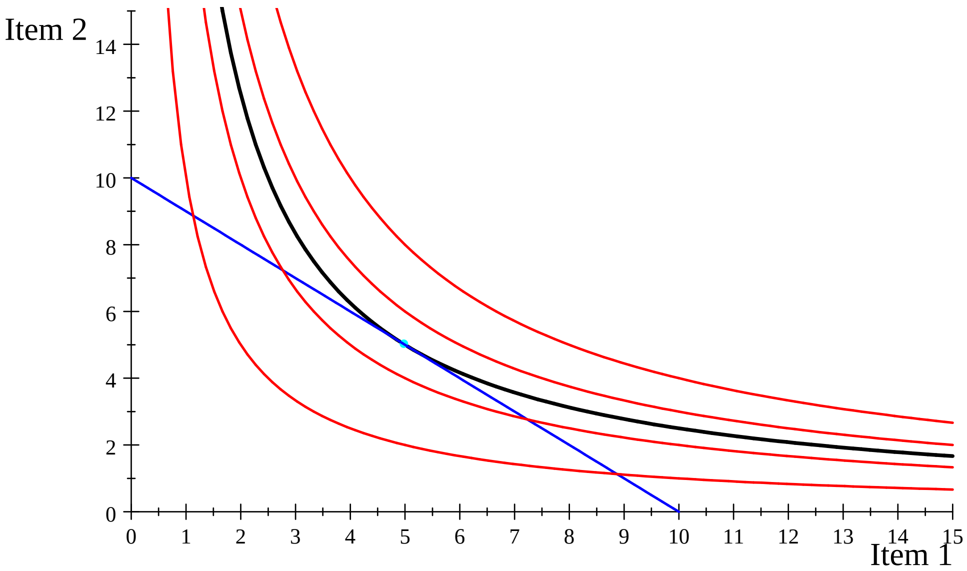
Definition 13 *Suppose a consumer has a well-behaved preference. The* ***marginal rate of substitution****, MRS, at the bundle* (x_1, x_2) *is the slope of the indifference curve at the point* (x_1, x_2)*: the rate at which the consumer is willing to substitute a little more consumption of item 2 (item 1) for a little less of consumption of item 1 (item 2):*

$$MRS = \frac{\Delta x_2}{\Delta x_1} \text{ (discrete change)} \qquad MRS = \frac{dx_2}{dx_1} \text{ (infinitesimal change)}$$

Remark 6 *If a consumer has a well-behaved preference, then the MRS is negative*



Remark 7 *Suppose that the consumer has a well-behaved. Suppose that the consumer owns the bundle $(5,5)$. Suppose that we offer him the following trade: he can exchange item 1 for item 2 in any amount at a "rate of exchange of E.", that is, $dx_2 = E dx_1$ or $dx_1 = \frac{dx_2}{E}$. Then, when the $MRS = -E$ the consumer is just on the margin of trading or not trading.*

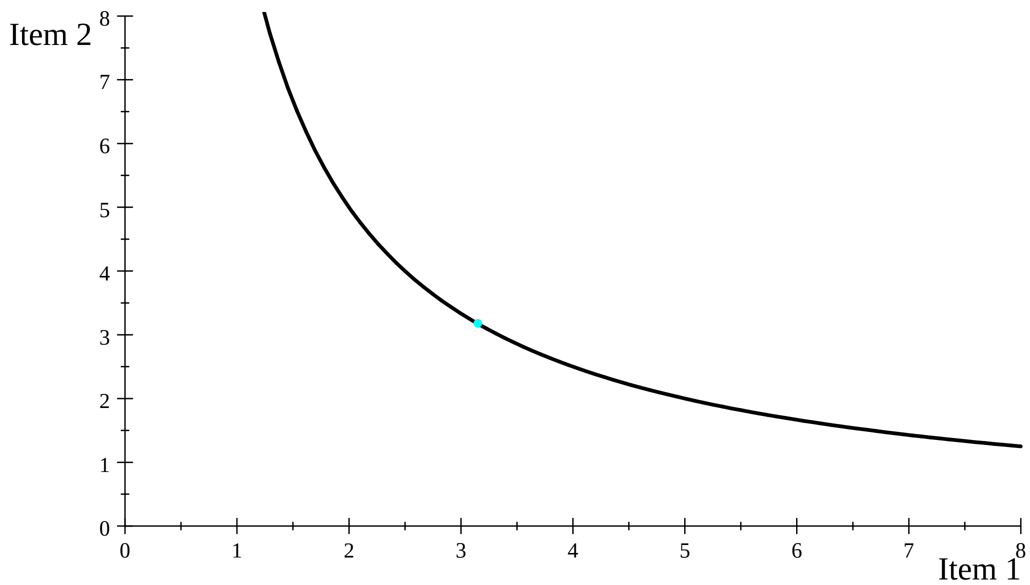


Remark 8 *If item 2 is money, then the MRS, that is, the rate at which the consumer substitutes item 2 for item 1, measures the* ***marginal willingness to pay*** *for item 1*

Definition 14 *The consumer's preference is **strictly convex** if for every two different bundles* (x_1, x_2) *and* (y_1, y_2) *and every number* t *such that* $0 < t < 1$*, it holds that*

$$(x_1, x_2) \sim (y_1, y_2) \implies (tx_1 + (1-t)\,y_1, tx_2 + (1-t)\,y_2) \succ (x_1, x_2)$$

Remark 9 *If a consumer has a strictly convex preference, then the consumer has strictly convex indifference curves. Moreover, the MRS decreases (in absolute value) as we increase* x_1*. We say that indifference curves exhibit a **diminishing MRS***



4 Utility

Definition 15 *Let $\succsim$ be the preference of the consumer. A real-valued function $u(\cdot)$ represents the consumer's preference $\succsim$ if and only if it holds that for every two bundles (x_1, x_2) and (y_1, y_2)*

$$(x_1, x_2) \succ (y_1, y_2) \iff u(x_1, x_2) > u(y_1, y_2)$$

$$(x_1, x_2) \sim (y_1, y_2) \iff u(x_1, x_2) = u(y_1, y_2)$$

If $u(\cdot)$ represents the consumer's preference, we say that $u(\cdot)$ is a (ordinal) ***utility function****.*

Example 6 *Suppose that there are only three bundles: $(2, 1)$, $(1, 2)$ and $(2, 2)$. Suppose that the consumer's preference is such that*

$$(2, 2) \succ (2, 1) \sim (1, 2).$$

Then, the real-valued function $u(\cdot)$ represents his preference if and only if

$$u(2, 2) > u(2, 1) = u(1, 2).$$

Remark 10 *The size of the utility difference does not matter. Only the order matters!!!*

Definition 16 *A consumer has a* ***Cobb-Douglas preference*** *if his preference* $\succsim$ *is represented by utility function of the form*

$$u(x_1, x_2) = x_1^a x_2^b, \quad \text{with } a > 0 \text{ and } b > 0$$

Definition 17 *A consumer has a* ***perfect-substitute preference*** *if his preference* $\succsim$ *is represented by utility function of the form*

$$u(x_1, x_2) = ax_1 + bx_2, \quad \text{with } a > 0 \text{ and } b > 0$$

Definition 18 *A consumer has a* ***perfect-complement preference*** *if his preference* $\succsim$ *is represented by utility function of the form*

$$u(x_1, x_2) = \min\{ax_1, bx_2\}, \quad \text{with } a > 0 \text{ and } b > 0$$

Definition 19 *A consumer has a* ***quasi-linear preference*** *(in item 2) if his preference* $\succsim$ *is represented by utility function of the form*

$$u(x_1, x_2) = v(x_1) + x_2$$

where the function $v(\cdot)$ *is an increasing and concave function. Typically,* $v(\cdot) = \ln(\cdot)$ *or* $v(\cdot) = \sqrt{\cdot}$.

***Definition 20** A **monotonic transformation** is a real-valued function $f : \mathbb{R} \to \mathbb{R}$ that transforms numbers into numbers in a way that preserves the order of the numbers, that is, for every two numbers r and s it hods that*

$$s > r \implies f(s) > f(r)$$

***Definition 21** Let the consumer's preference $\succsim$ be represented by $u(\cdot)$. If $f(u(\cdot))$ is a monotonic transformation of $u(\cdot)$, then $f(u(\cdot))$ is also a utility function that represents the consumer's preference $\succsim$. That is, it holds that*

$$(x_1, x_2) \succ (y_1, y_2) \iff f(u(x_1, x_2)) > f(u(y_1, y_2))$$

Example 7 *Suppose that Alice's utility function is $u_A(x_1, x_2) = x_1 x_2$. Bob's utility function is $u_B(x_1, x_2) = 1000x_1x_2 + 2000$, whereas Caroline's utility function is $u_C(x_1, x_2) = -x_1x_2$.*

Who has the same preference as Alice?

Only Bob as the same preference of Alice since for every two bundles (x_1, x_2) and (y_1, y_2), it holds that

$$u_A(x_1, x_2) > u_A(y_1, y_2) \iff u_B(x_1, x_2) > u_B(y_1, y_2)$$

where

$$u_B(\cdot) = f(u_A(\cdot)) = 1000u_A(\cdot) + 2000$$

The utility function of Caroline is not a monotonic transformation of Alice's given that

$$u_C(\cdot) = -u_A(\cdot)$$

(Caroline's order of bundles is reversed).

Example 8 *Suppose that Alice's utility function is* $u_A(x_1, x_2) = x_1^a x_2^b$ *where* a *and* b *are positive numbers. Bob's utility function is* $u_B(x_1, x_2) = a \ln x_1 + b \ln x_2$*. Does Bob have the same preference as Alice?*

The answer is yes since for every two bundles (x_1, x_2) *and* (y_1, y_2)*, it holds that*

$$u_A(x_1, x_2) > u_A(y_1, y_2) \iff u_B(x_1, x_2) > u_B(y_1, y_2)$$

where

$$u_B(\cdot) = \ln(u_A(\cdot)),$$

that is, $u_B(\cdot)$ *is a logarithm (and so monotonic) transformation of* $u_A(\cdot)$

Definition 22 *Suppose that the consumer's preference $\succsim$ over consumption bundles is represented by the utility function $u(\cdot)$. Take any bundle (x_1, x_2). The* ***indifference curve at the utility level*** $u(x_1, x_2) = k$*, denoted by* $IC(k)$*, consists of bundles that give a level of utility equal to* k*, that is,*

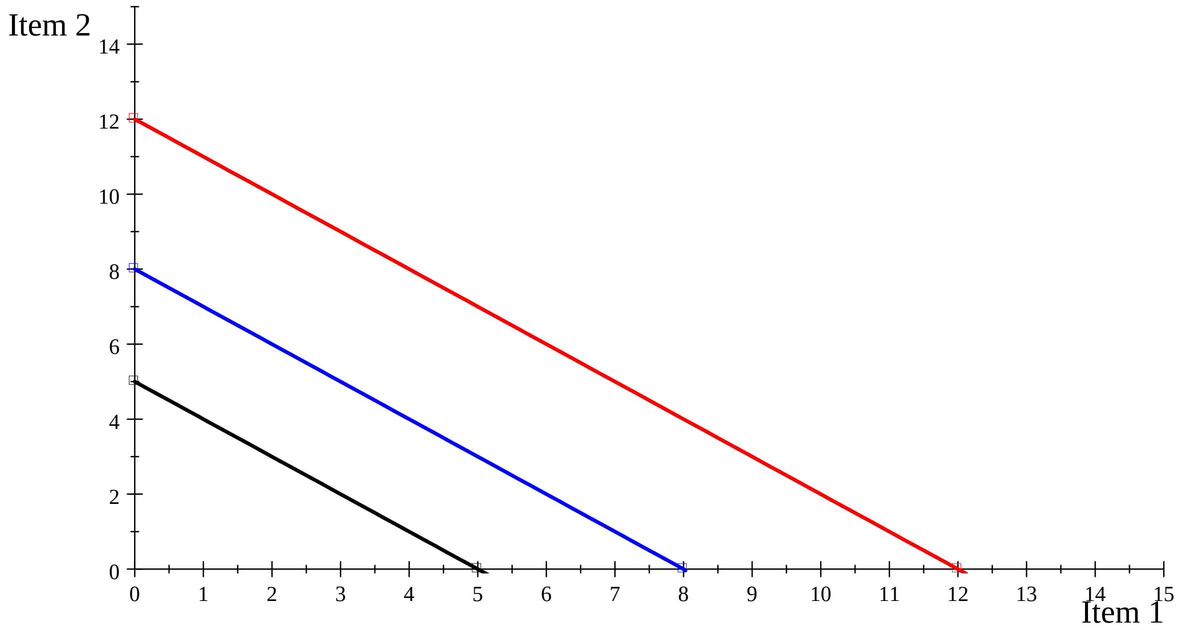
$$IC(k) = \{(y_1, y_2) \, | u(x_1, x_2) = u(y_1, y_2) = k\}$$

Example 9 *Suppose that the consumer has a perfect-substitute preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = ax_1 + bx_2, \quad \textit{with } a > 0 \textit{ and } b > 0$$

An indifference curve at level k has the formula

$$x_2 = \frac{k}{b} - \frac{a}{b}x_1$$



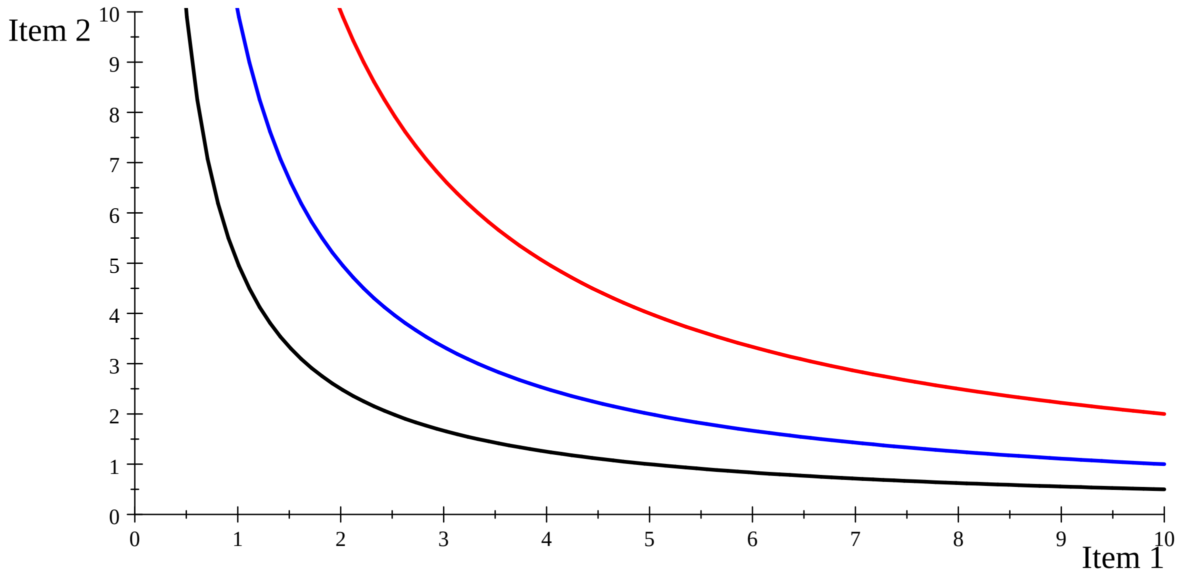
$a = b = 1$ and $k = 5, 8, 12$

Example 10 *Suppose that the consumer has a Cobb-Douglas preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = x_1^a x_2^b, \quad \text{with } a > 0 \text{ and } b > 0$$

An indifference curve at level k has the formula

$$x_2 = \left(\frac{k}{x_1^a}\right)^{\frac{1}{b}}$$



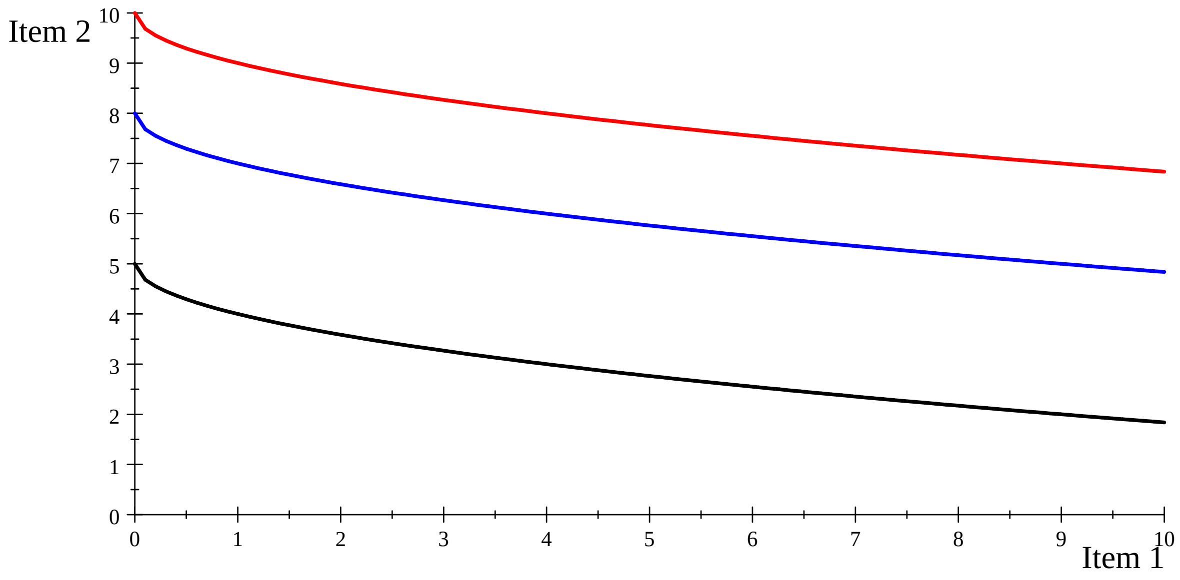
$a = b = 1$ and $k = 5, 10, 20$

Example 11 *Suppose that the consumer has a quasi-linear preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = v(x_1) + x_2$$

An indifference curve at level k has the formula

$$x_2 = k - v(x_1)$$



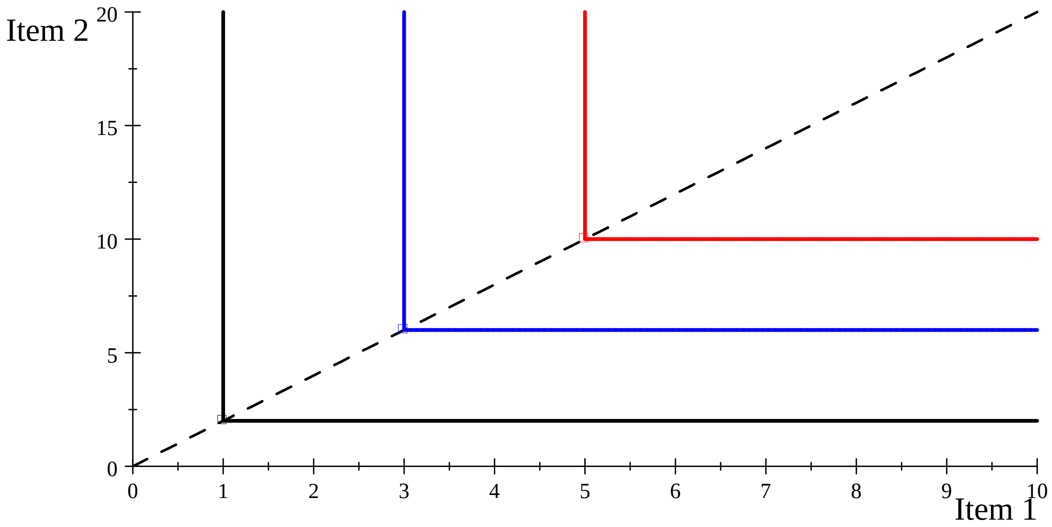
$v(\cdot) = \sqrt{\cdot}$ and $k = 5, 8, 10$

Example 12 *Suppose that the consumer has a perfect-complement preference, that is, his preference is represented by the utility function* $u(x_1, x_2) = \min\{ax_1, bx_2\}$, *with* $a > 0$ *and* $b > 0$
The consumer likes to consume item 1 and 2 in a fixed proportion equal to

$$ax_1 = bx_2 \implies x_2 = \left(\frac{a}{b}\right) x_1$$

An indifference curve at level k *has the formula*

$$ax_1 \leq bx_2 \implies x_1 = \frac{k}{a}; \quad ax_1 \geq bx_2 \implies x_2 = \frac{k}{b}$$



$a = 1, b = 0.5$ and $k = 1, 3, 5$

Definition 23 *Suppose that the utility function $u(\cdot)$ represents the consumer's preference $\succsim$. The* ***marginal utility of item*** *$i = 1, 2$ is the rate at which the consumer's utility increases as the consumption of item $i = 1, 2$ increases, while we hold the quantity of the other item constant.*
Formally, let Δx_i represent an increment of item $i = 1, 2$. The marginal utility of item $i = 1, 2$, denoted by MU_i, is defined as

$$\text{for } i = 1 : MU_1 = \lim_{\Delta x_1 \to 0} \left(\frac{u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)}{\Delta x_1} \right) = \frac{\partial u(x_1, x_2)}{\partial x_1}$$

$$\text{for } i = 2 : MU_2 = \lim_{\Delta x_2 \to 0} \left(\frac{u(x_1, x_2 + \Delta x_2) - u(x_1, x_2)}{\Delta x_2} \right) = \frac{\partial u(x_1, x_2)}{\partial x_2}$$

Remark 11 *The derivative of a function of two variables $u(x_1, x_2)$, with respect to x_1 while x_2 being held constant, is called the partial derivative of the function $u(x_1, x_2)$ with respect to x_1. A partial derivative is commonly shown with a ∂ symbol instead of a d.*

Remark 12 *Suppose that the consumer has a perfect-substitute preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = ax_1 + bx_2, \quad \textit{with } a > 0 \textit{ and } b > 0$$

Then,

$$MU_1 = a \quad \textit{and} \quad MU_2 = b$$

Remark 13 *Suppose that the consumer has a Cobb-Douglas preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = x_1^a x_2^b, \quad \textit{with } a > 0 \textit{ and } b > 0$$

Then,

$$MU_1 = ax_1^{a-1}x_2^b \quad \textit{and} \quad MU_2 = bx_1^a x_2^{b-1}$$

***Remark 14** Suppose that the consumer has a quasi-linear preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = v(x_1) + x_2$$

Then,

$$MU_1 = \frac{dv(x_1)}{dx_1} \quad \textit{and} \quad MU_2 = 1$$

Note that:

$$v(x_1) = \ln x_1 \implies \frac{dv(x_1)}{dx_1} = \frac{1}{x_1}$$

and that:

$$v(x_1) = \sqrt{x_1} \implies \frac{dv(x_1)}{dx_1} = \frac{1}{2}\frac{1}{\sqrt{x_1}}$$

Remark 15 *Suppose that the consumer has a perfect-complement preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = \min\{ax_1, bx_2\} \quad \textit{with } a > 0 \textit{ and } b > 0$$

Then, we need to distinguish three cases:

Case 1: $ax_1 > bx_2 \implies MU_1 = 0$ *and* $MU_2 = b$

Case 2: $ax_1 < bx_2 \implies MU_1 = a$ *and* $MU_2 = 0$

Case 3: $ax_1 = bx_2 \implies MU_i$ *is undefined for* $i = 1, 2$

Remark 16 (IMPORTANT!) *Suppose that the utility function $u(\cdot)$ represents the consumer's preference $\succsim$. The indifference curve at the utility level consists of bundles that give a level of utility equal to k, that is,*

$$IC(k) = \{(y_1, y_2) \, | u(y_1, y_2) = k\}$$

Suppose that the bundles (x_1, x_2) and (x'_1, x'_2) are elements of $IC(k)$, that is,

$$u(x_1, x_2) = u(x'_1, x'_2) = k$$

Denote $\Delta x_i = (x'_i - x_i)$ the change in the consumption of item $i = 1, 2$

The total change in utility is

$$MU_1 \Delta x_1 + MU_2 \Delta x_2 = 0$$

From this we have that

$$MRS = \frac{\Delta x_2}{\Delta x_1} = -\frac{MU_1}{MU_2}$$

This gives us a convenient tool for calculating the MRS, either as a function of (x_1, x_2) or as a numerical value at a given point

Remark 17 *Suppose that the consumer has a perfect-substitute preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = ax_1 + bx_2, \quad \text{with } a > 0 \text{ and } b > 0$$

Then,

$$MU_1 = a, \quad MU_2 = b \quad \text{and} \quad MRS = -\frac{a}{b}$$

Remark 18 *Suppose that the consumer has a Cobb-Douglas preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = x_1^a x_2^b, \quad \text{with } a > 0 \text{ and } b > 0$$

Then,

$$MU_1 = ax_1^{a-1}x_2^b, \quad MU_2 = bx_1^a x_2^{b-1} \quad \text{and} \quad MRS = -\frac{a}{b}\frac{x_2}{x_1}$$

Remark 19 *Suppose that the consumer has a quasi-linear preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = v(x_1) + x_2$$

Then,

$$MU_1 = \frac{dv(x_1)}{dx_1}, \quad MU_2 = 1 \quad \text{and} \quad MRS = -\frac{dv(x_1)}{dx_1}$$

Note that:

$$v(x_1) = \ln x_1 \implies \frac{dv(x_1)}{dx_1} = \frac{1}{x_1}$$

and that:

$$v(x_1) = \sqrt{x_1} \implies \frac{dv(x_1)}{dx_1} = \frac{1}{2}\frac{1}{\sqrt{x_1}}$$

Remark 20 *Suppose that the consumer has a perfect-complement preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = \min\{ax_1, bx_2\} \qquad \textit{with } a > 0 \textit{ and } b > 0$$

Then, we need to distinguish three cases:

Case 1: $ax_1 > bx_2 \implies MU_1 = 0$, $MU_2 = b$ *and* $MRS = 0$

Case 2: $ax_1 < bx_2 \implies MU_1 = a$, $MU_2 = 0$ *and* $MRS = \infty$

Case 3: $ax_1 = bx_2 \implies$ *the* MU_i *for* $i = 1, 2$ *and* MRS *are undefined*

5 Choice

Problem 1 *Suppose that the utility function $u(\cdot)$ represents the consumer's preference $\succsim$. Suppose that p_i is the price of item $i = 1, 2$. Suppose that the consumer's income is m.*
The consumer optimal's consumption bundle (x_1^*, x_2^*) is the solution to this ***constraint maximization problem****:*

$$\underset{x_1, x_2}{\textit{Maximize}} \quad u(x_1, x_2)$$

subject to (the budget constraint)

$$p_1 x_1 + p_2 x_2 = m$$

Assumption 2 *We assume that the consumer spends his income entirely*

- We can solve the above problem by using one of the following three methods

Method 1 (Brute force method) *This method, as the name suggests, may be rather ugly and difficult.*

Step 1. *Use the constraint to solve for* x_2*:*

$$x_2(x_1) = \frac{m}{p_2} - \frac{p_1}{p_2}x_1$$

Step 2. *Plug* $x_2(x_1)$ *into* $u(x_1, x_2)$ *which gives*

$$u\left(x_1, \frac{m}{p_2} - \frac{p_1}{p_2}x_1\right)$$

Step 3. *Differentiate this function (of one variable) with respect to* x_1 *and set the result equal to zero, that is,*

$$\frac{du\left(x_1, \frac{m}{p_2} - \frac{p_1}{p_2}x_1\right)}{dx_1} = 0$$

Solving the resulting equation with respect to x_1 *gives* x_1^*

Step 4. *Plug* x_1^* *into* $x_2(x_1)$ *which gives*

$$x_2^* = \frac{m}{p_2} - \frac{p_1}{p_2}x_1^*$$

Method 2 (Use-the-graphs method) *We can rely on what we learned so far from graphs.*

Step 1. *The optimal consumption choice is where an indifference curve is tangent to the budget line, that is, where*

$$MRS = -\frac{MU_1}{MU_2} = -\frac{p_1}{p_2}$$

Step 2. *Combine the above equation with the budget constraint equation to solve for the two unknowns* (x_1^*, x_2^*)

Remark 21 *This is the method we use most often in these lectures*

Method 3 (The Lagrange function method) *The standard mathematical method for solving a constrained maximization problem is the following:*

Step 1. *Set up a special function L, called the* ***Lagrange function****, that incorporates both the utility function and the budget constraint,*

$$L = u(x_1, x_2) + \lambda (m - (p_1 x_1 + p_2 x_2))$$

where λ is a special variable called the ***Lagrange multiplier***

Step 2. *Find the first-order conditions for the maximization of L with respect to x_1, x_2 and λ which result in a system of three equations in three unknowns*

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \lambda} = 0$$

Step 3. *Solve for the optimal quantities of the items (x_1^*, x_2^*) and for the optimal λ^*.*

Remark 22 *The Lagrange multiplier has a nice economic interpretation in terms of how much the consumer would value a one-pound increase in his income. This method is more elegant that the previous ones and will be taught in ME.*

Example 13 *Suppose that the consumer has a Cobb-Douglas preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = x_1^a x_2^b, \quad \text{with } a > 0 \text{ and } b > 0$$

The tangency condition leads to

$$MRS = -\frac{a}{b}\frac{x_2}{x_1} = -\frac{p_1}{p_2} \iff ap_2x_2 = bp_1x_1$$

It follows that

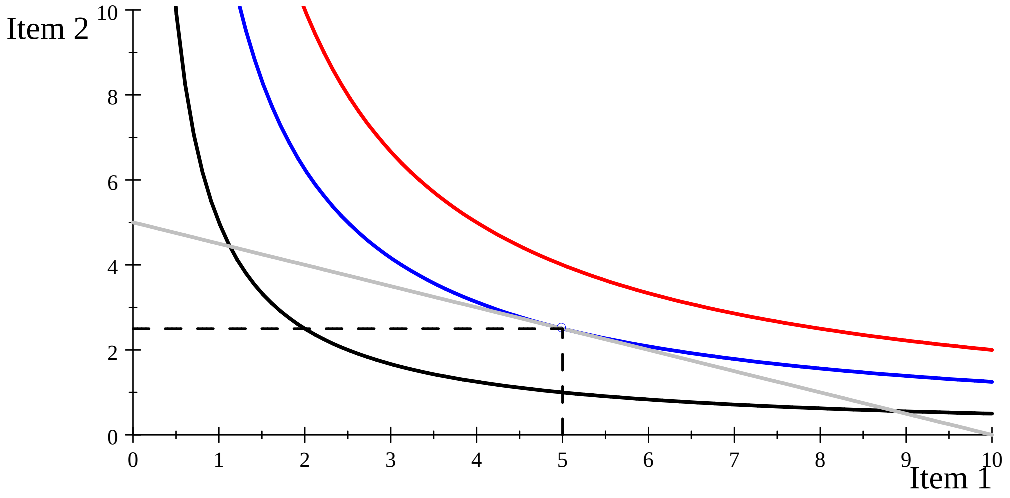
$$p_2x_2 = \frac{b}{a}p_1x_1 \quad \text{or} \quad \frac{a}{b}p_2x_2 = p_1x_1$$

Plug $p_2x_2 = \frac{b}{a}p_1x_1$ *into the budget constraint equation* $p_1x_1 + p_2x_2 = m$ *gives*

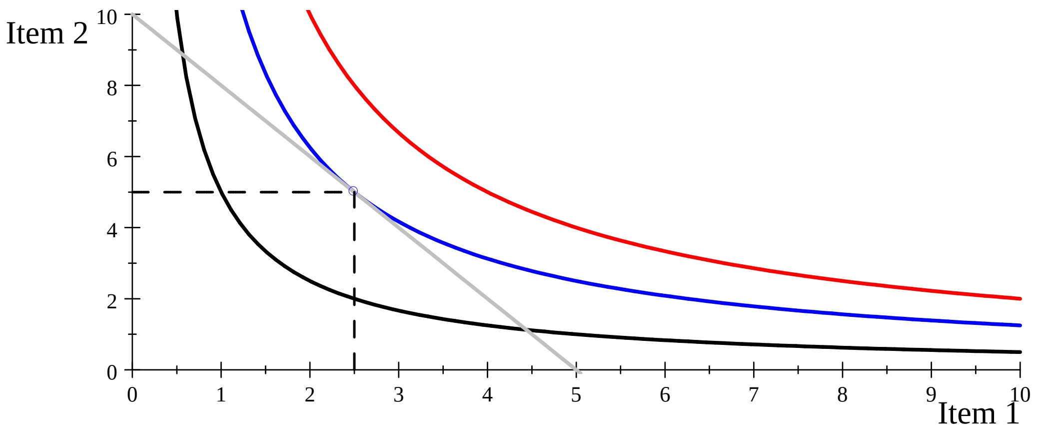
$$x_1^* = \left(\frac{a}{a+b}\right)\frac{m}{p_1}$$

Plug $\frac{a}{b}p_2x_2 = p_1x_1$ *into the budget constraint equation* $p_1x_1 + p_2x_2 = m$ *gives*

$$x_2^* = \left(\frac{b}{a+b}\right)\frac{m}{p_2}$$



$p_1 = 1$, $p_2 = 2$, $m = 10$, $a = b = 1$ and $k = 5, 12.5, 20$



$p_1 = 2$, $p_2 = 1$, $m = 10$, $a = b = 1$ and $k = 5, 12.5, 20$

Example 14 *Suppose that the consumer has a perfect-substitute preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = ax_1 + bx_2, \quad \textit{with } a > 0 \textit{ and } b > 0$$

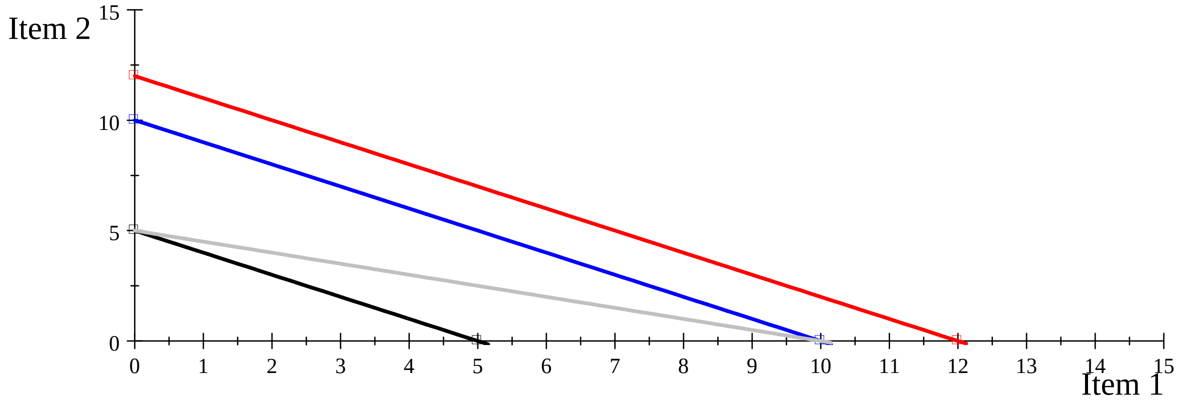
The indifference curves are straight lines with slope equal to $MRS = -\frac{a}{b}$.

The optimal choice for item 1 is

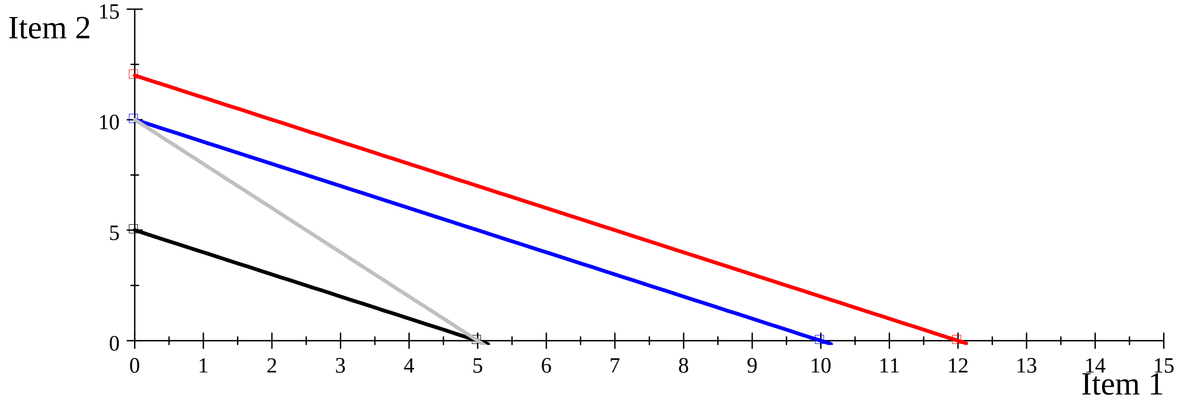
$$x_1^* = \begin{cases} \frac{m}{p_1} & \textit{if } p_1 < p_2 \\ \textit{any number between 0 and } \frac{m}{p_1} & \textit{if } p_1 = p_2 \\ 0 & \textit{if } p_1 > p_2 \end{cases}$$

The optimal choice for item 2 is

$$x_2^* = \begin{cases} 0 & \textit{if } p_1 < p_2 \\ \textit{any number between 0 and } \frac{m}{p_1} & \textit{if } p_1 = p_2 \\ \frac{m}{p_2} & \textit{if } p_1 > p_2 \end{cases}$$



$a = b = 1$, $p_1 = 1$, $p_2 = 2$, $m = 10$ and $k = 5, 10, 12$



$a = b = 1$, $p_1 = 2$, $p_2 = 1$, $m = 10$ and $k = 5, 10, 12$

Example 15 *Suppose that the consumer has a quasi-linear preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = \sqrt{x_1} + x_2$$

The tangency condition leads to

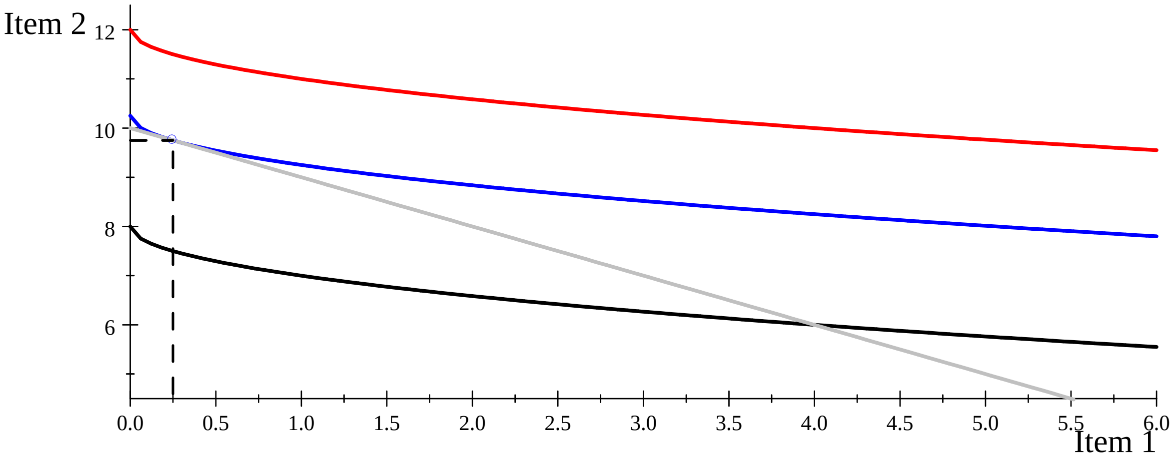
$$MRS = -\frac{1}{2}\frac{1}{\sqrt{x_1}} = -\frac{p_1}{p_2}$$

The optimal choice for item 1 is

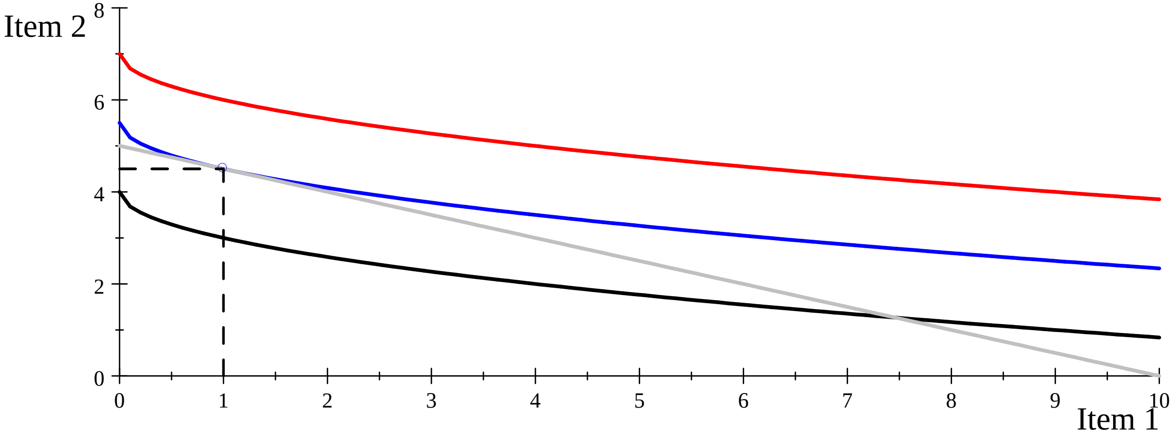
$$x_1^* = \frac{1}{4}\left(\frac{p_2}{p_1}\right)^2$$

Plug x_1^* into the budget constraint equation $p_1x_1 + p_2x_2 = m$ gives

$$x_2^* = \frac{m}{p_2} - \frac{p_2}{4p_1}$$



$p_1 = p_2 = 1$, $m = 10$ and $k = 8, 10.25, 12$



$p_1 = 1$, $p_2 = 2$, $m = 10$ and $k = 4, 5.5, 10$

Example 16 *Suppose that the consumer has a perfect-complement preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = \min\{ax_1, bx_2\} \qquad \textit{with } a > 0 \textit{ and } b > 0$$

The consumer likes to consume item 1 and 2 in a fixed proportion equal to

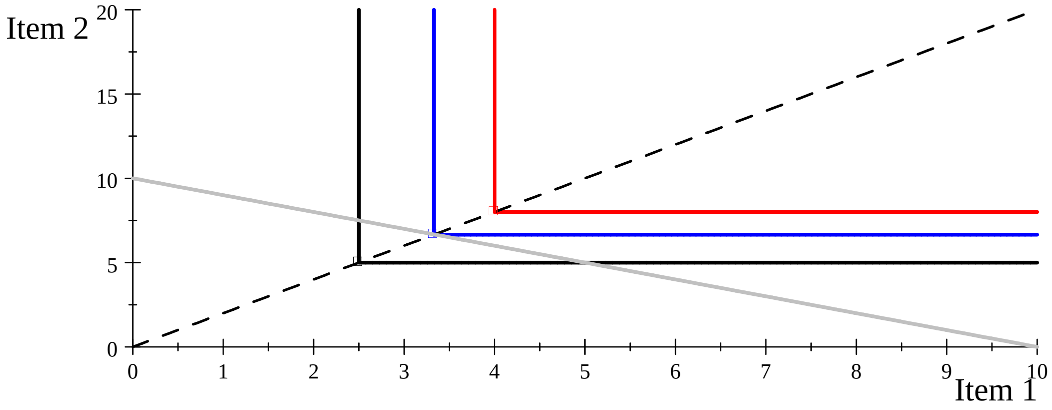
$$ax_1 = bx_2 \iff x_2 = \left(\frac{a}{b}\right) x_1 \quad \textit{or} \quad x_1 = \left(\frac{b}{a}\right) x_2$$

Plug $x_2 = \left(\frac{a}{b}\right) x_1$ *into the budget constraint equation* $p_1x_1 + p_2x_2 = m$ *gives*

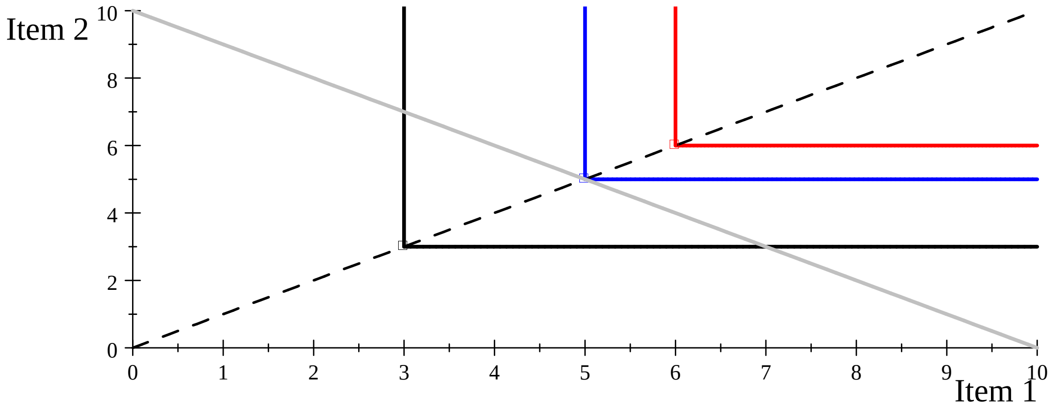
$$x_1^* = b\left(\frac{m}{bp_1 + ap_2}\right)$$

Plug $x_1 = \left(\frac{b}{a}\right) x_2$ *into the budget constraint equation* $p_1x_1 + p_2x_2 = m$ *gives*

$$x_2^* = a\left(\frac{m}{bp_1 + ap_2}\right)$$



$p_1 = p_2 = 1$, $m = 10$, $a = 1, b = 0.5$ and $k = 2.5, 3.33, 4$



$p_1 = p_2 = 1$, $m = 10$, $a = b = 1$ and $k = 3, 5, 6$

6 Individual Demand

Problem 2 *Suppose that the utility function* $u(\cdot)$ *represents the consumer's preference* $\succsim$*. Suppose that* p_i *is the price of item* $i = 1, 2$*. Suppose that the consumer's income is* m*.*
The consumer optimal's consumption bundle (x_1^*, x_2^*) *is the solution to this* ***constraint maximization problem****:*

$$\underset{x_1, x_2}{\text{Maximize}} \quad u(x_1, x_2)$$

subject to (the budget constraint)

$$p_1 x_1 + p_2 x_2 = m$$

Remark 23 *Because the utility-maximizing bundle* (x_1^*, x_2^*) *can be made for any value of* p_1*,* p_2 *and* m*, we are actually finding two demand functions:*

$$x_1^* = x_1(p_1, p_2, m) \qquad \text{and} \qquad x_2^* = x_2(p_1, p_2, m)$$

where $x_i(p_1, p_2, m)$ *is the consumer's demand function for item* $i = 1, 2$

Definition 24 *An item $i = 1, 2$ is said to be* ***normal*** *if the quantity demanded for it increases as income increases,*

$$\frac{dx_i(p_1, p_2, m)}{dm} > 0$$

Definition 25 *An item $i = 1, 2$ is said to be* ***inferior*** *if an increase in income results in a reduction in its consumption*

Definition 26 *The curve that connects the different optimal choices of the consumer for different income levels is called the* ***income offer curve*** *(or* ***income expansion path****)*

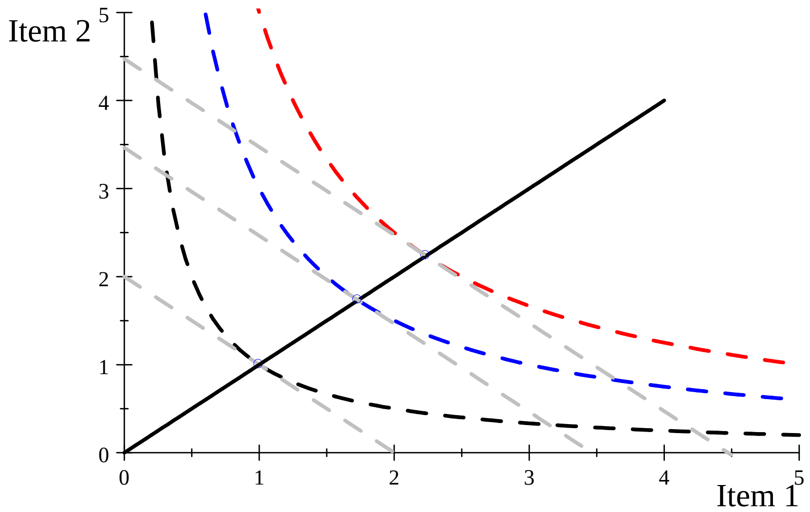
Definition 27 *The curve that connects the consumer's demand for item $i = 1, 2$ for different income levels is called the* ***Engel curve***

Example 17 *Suppose that the consumer has a Cobb-Douglas preference, that is, his preference is represented by the utility function*

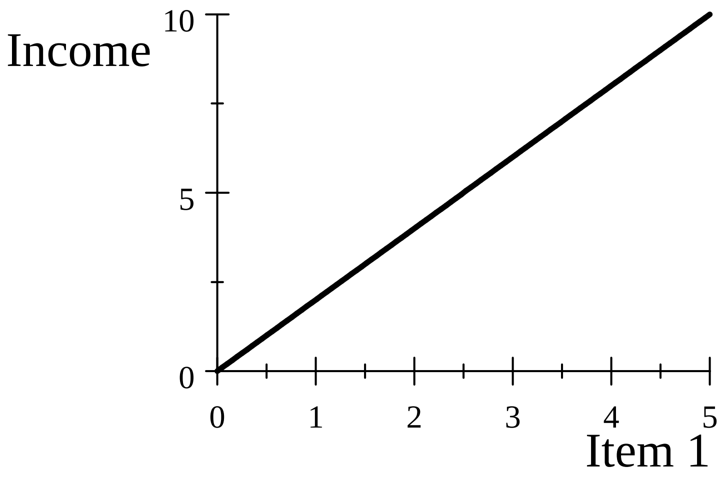
$$u(x_1, x_2) = x_1^a x_2^b, \quad \textit{with } a = b = 1$$

Suppose that $p_i = 1$ *for* $i = 1, 2$*. The consumer's demand functions are*

$$x_1(p_1, p_2, m) = \frac{m}{2} \quad \textit{and} \quad x_2(p_1, p_2, m) = \frac{m}{2}$$



Income offer curve, $m = 2, 2\sqrt{3}, 2\sqrt{5}$



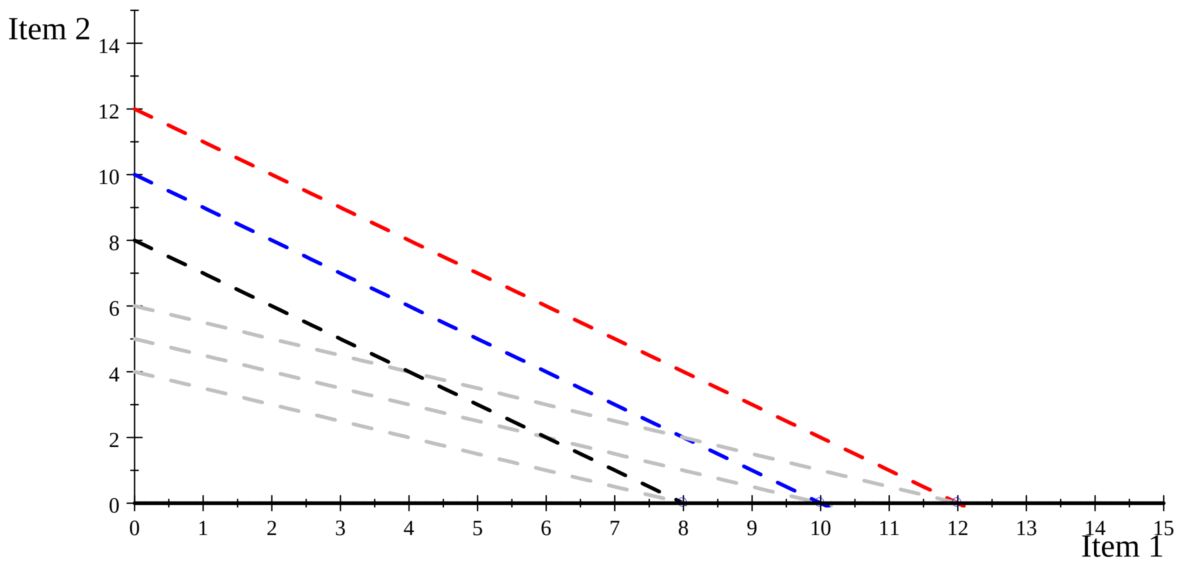
Engel curve for item 1, slope $\frac{p_1(a+b)}{a}$

Example 18 *Suppose that the consumer has a perfect-substitute preference, that is, his preference is represented by the utility function*

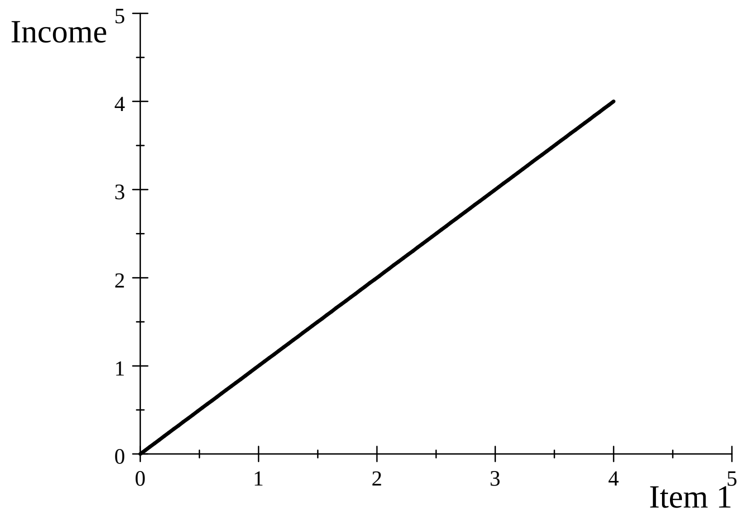
$$u(x_1, x_2) = ax_1 + bx_2, \quad \textit{with } a = b = 1$$

Suppose that $p_1 = 1 < p_2 = 2$*, so that the consumer specializes in the consumption of item 1. The consumer's demand functions are*

$$x_1(p_1, p_2, m) = m \quad \textit{and} \quad x_2(p_1, p_2, m) = 0$$



Income offer curve, $m = 8, 10, 12$



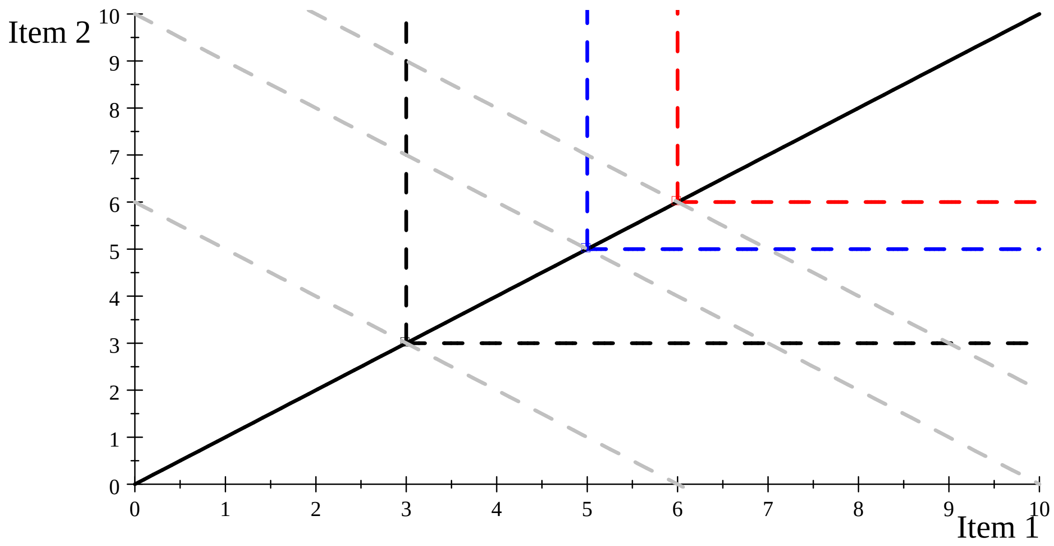
Engel curve for item 1, slope p_1

Example 19 *Suppose that the consumer has a perfect-complement preference, that is, his preference is represented by the utility function*

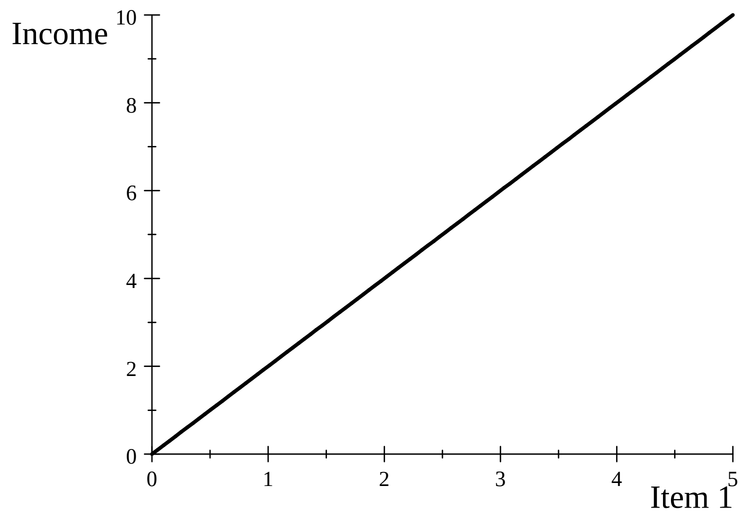
$$u(x_1, x_2) = \min\{ax_1, bx_2\} \quad \textit{with } a = b = 1$$

Suppose that $p_i = 1$ *for* $i = 1, 2$*. The consumer's demand functions are*

$$x_1(p_1, p_2, m) = \left(\frac{m}{p_1 + p_2}\right) \quad \textit{and} \quad x_2(p_1, p_2, m) = \left(\frac{m}{p_1 + p_2}\right)$$



Income offer curve, $m = 6, 10, 12$



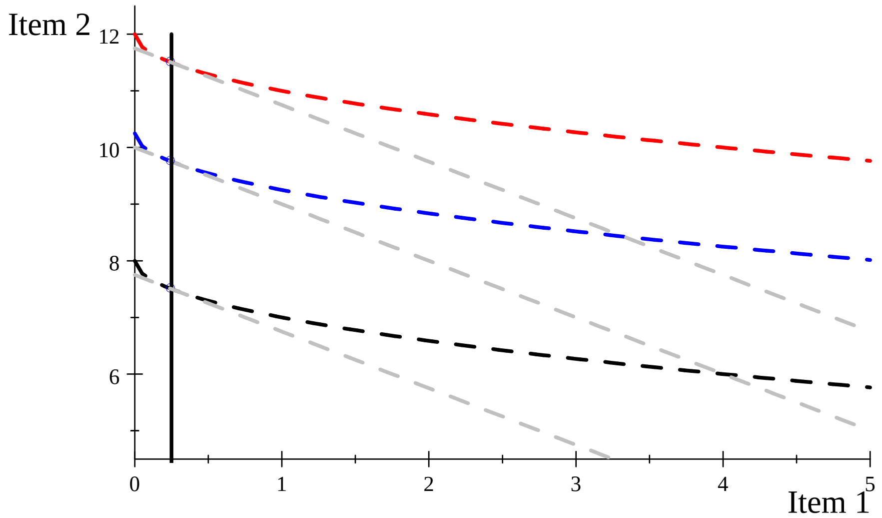
Engel curve for item 1, slope $(p_1 + p_2)$

Example 20 *Suppose that the consumer has a quasi-linear preference, that is, his preference is represented by the utility function*

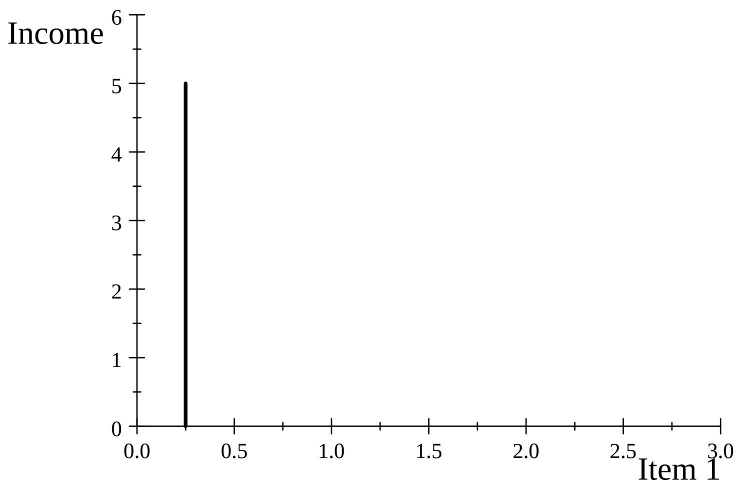
$$u(x_1, x_2) = \sqrt{x_1} + x_2$$

Suppose that $p_i = 1$ *for* $i = 1, 2$*. The consumer's demand functions are*

$$x_1(p_1, p_2, m) = \frac{1}{4} \quad \text{and} \quad x_2(p_1, p_2, m) = m - \frac{1}{4}$$



Income offer curve, $m = 7.75, 10, 11.75$



Engel curve for item 1

Definition 28 *An item $i = 1, 2$ is said to be **ordinary** if the quantity demanded for it increases as its own price decreases,*

$$\frac{dx_i\left(p_1, p_2, m\right)}{dp_i} < 0 \qquad \textit{(law of demand)}$$

Definition 29 *An item $i = 1, 2$ is said to be **Giffen** if an increase in its own price results in an increase in its consumption*

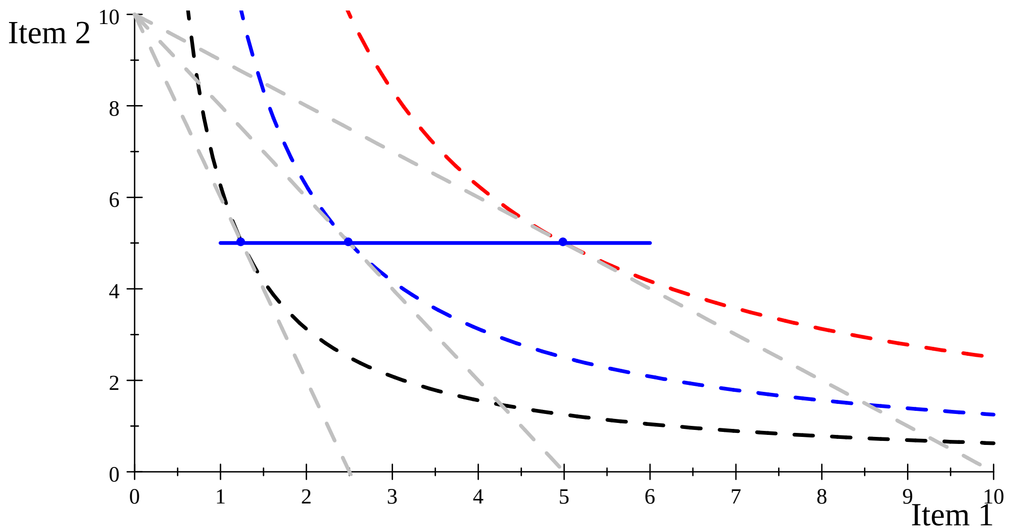
Definition 30 *The curve that connects the different optimal choices of the consumer for different prices of item $i = 1, 2$ is called the **price offer curve***

Example 21 *Suppose that the consumer has a Cobb-Douglas preference, that is, his preference is represented by the utility function*

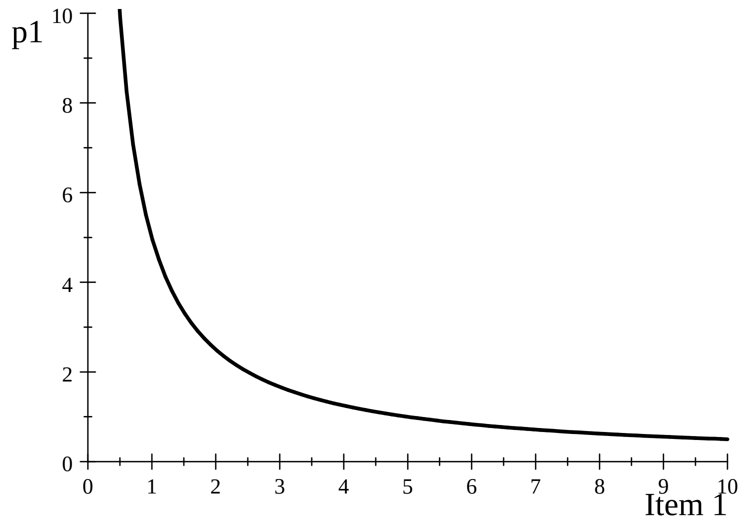
$$u(x_1, x_2) = x_1^a x_2^b, \quad \textit{with } a = b = 1$$

Suppose that $p_2 = 1$ *and* $m = 10$*. The consumer's demand function for item 1 is*

$$x_1(p_1, p_2, m) = \frac{5}{p_1}$$



Price offer curve, $p_1 = 1, 2, 4$



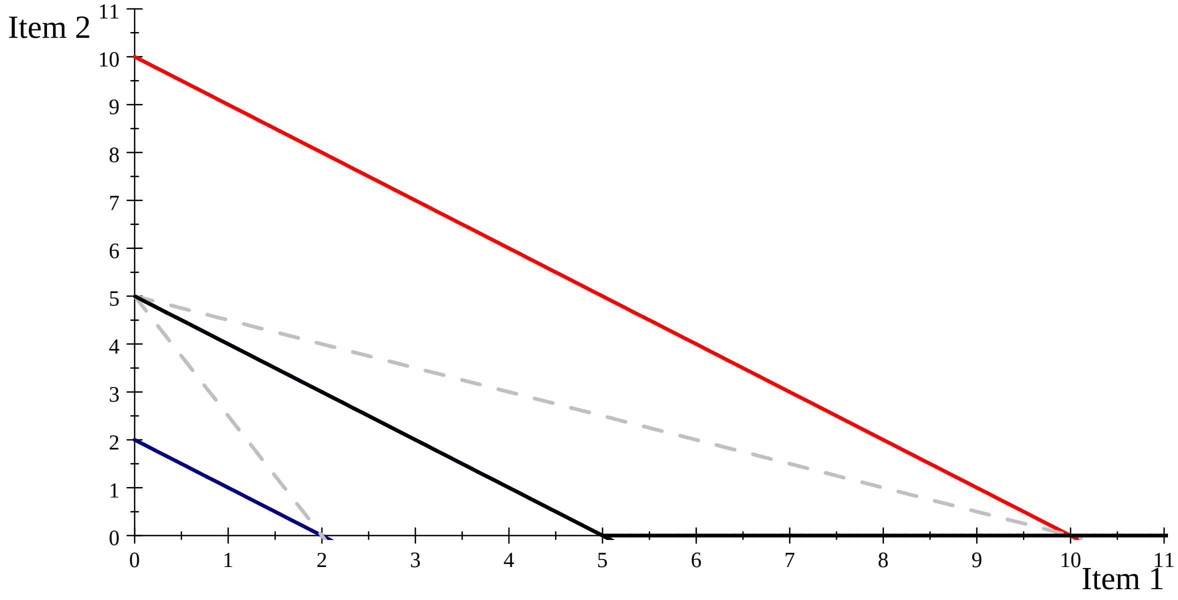
Demand curve for item 1

Example 22 *Suppose that the consumer has a perfect-substitute preference, that is, his preference is represented by the utility function*

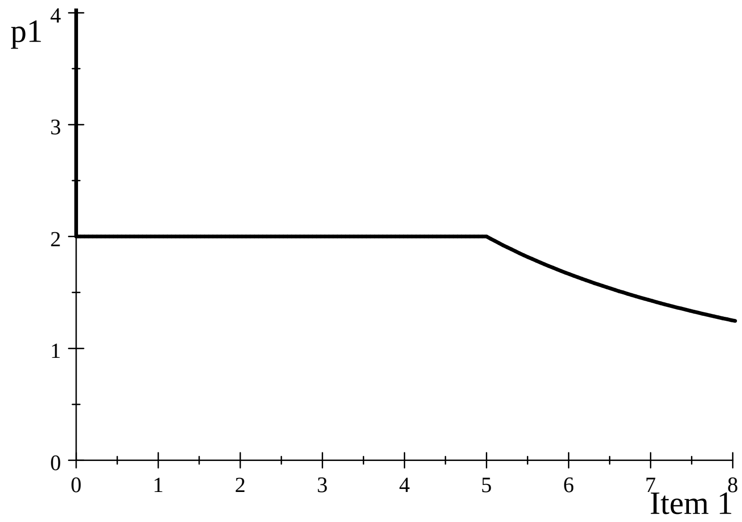
$$u(x_1, x_2) = ax_1 + bx_2, \quad \textit{with } a = b = 1$$

Suppose that $p_2 = 2$ *and* $m = 10$*. The consumer's demand function for item 1 is*

$$x_1(p_1, p_2, m) = \begin{cases} \frac{m}{p_1} & \textit{if } p_1 < p_2 \\ \textit{any number between 0 and } \frac{m}{p_1} & \textit{if } p_1 = p_2 \\ 0 & \textit{if } p_1 > p_2 \end{cases}$$



Price offer curve, $p_1 = 1, 2, 5$



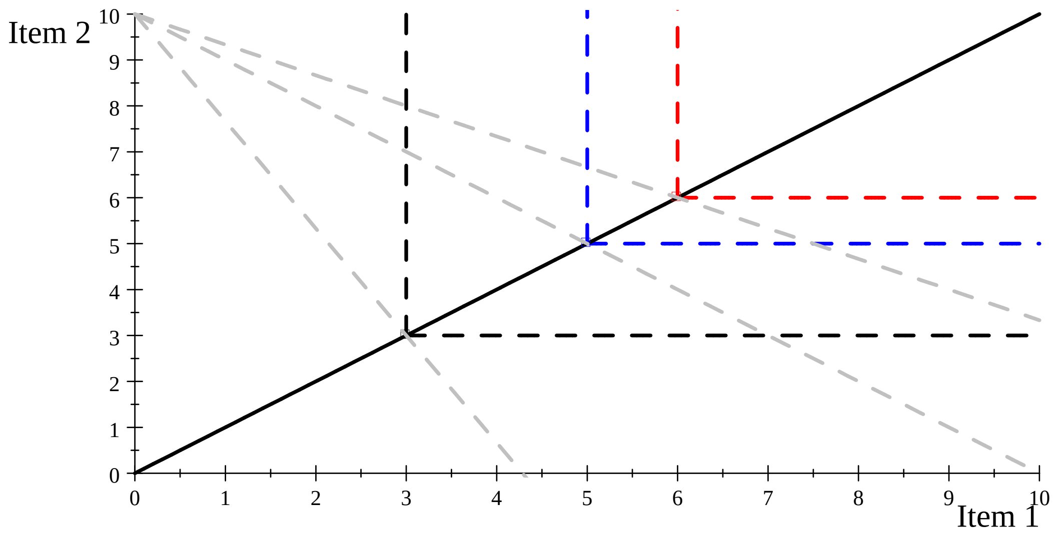
Demand curve for item 1

Example 23 *Suppose that the consumer has a perfect-complement preference, that is, his preference is represented by the utility function*

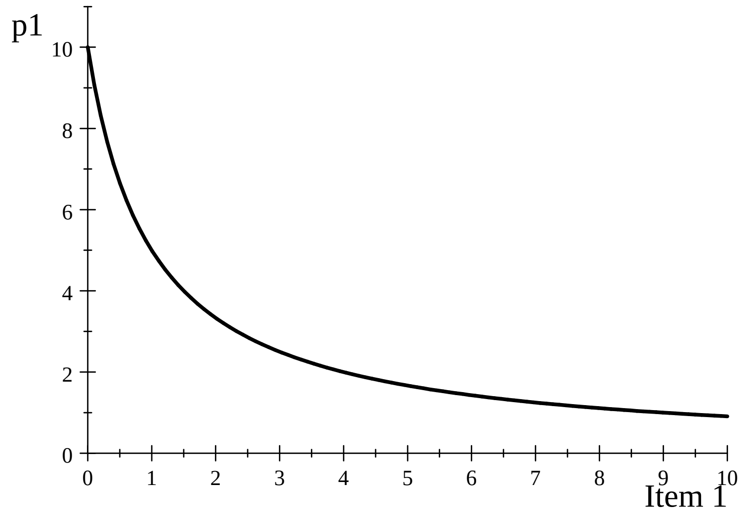
$$u(x_1, x_2) = \min\{ax_1, bx_2\} \qquad \textit{with } a = b = 1$$

Suppose that $p_2 = 1$ *and* $m = 10$ *The consumer's demand functions for item 1 is*

$$x_1(p_1, p_2, m) = \left(\frac{10}{p_1 + 1}\right)$$



Price offer curve, $p_1 = \frac{2}{3}, 1, \frac{7}{3}$



Demand curve for item 1

Definition 31 *If the demand of item 1 (item 2) goes up when the price of item 2 (item 1) goes up, then we say that item 1 (item 2) is a (**gross**) **substitute** for item 2 (item 1). In terms of rate of change, item 1 (item 2) is a substitute for item 2 (item 1) if*

$$\frac{dx_1\left(p_1,p_2,m\right)}{dp_2}>0\ \left(\frac{dx_2\left(p_1,p_2,m\right)}{dp_1}>0\right)$$

Definition 32 *If the demand of item 1 (item 2) goes down when the price of item 2 (item 1) goes up, then we say that item 1 (item 2) is a (**gross**) **complement** to item 2 (item 1). In terms of rate of change, item 1 (item 2) is a complement for item 2 (item 1) if*

$$\frac{dx_1\left(p_1,p_2,m\right)}{dp_2}<0\ \left(\frac{dx_2\left(p_1,p_2,m\right)}{dp_1}<0\right)$$

Definition 33 *For ordinary items we know that we have downward-sloping demand curves. That is, if item 1 is an ordinary item, then*

$$x_1 (p_1, p_2, m) \text{ with } \frac{dx_1 (p_1, p_2, m)}{dp_1} < 0.$$

The ***inverse demand function*** *for item 1, denoted by $p(x_1)$, is the demand function for item 1 viewing its price as a function of the quantity x_1*

Example 24 *Suppose that the consumer has a Cobb-Douglas preference, that is, his preference is represented by the utility function*

$$u (x_1, x_2) = x_1^a x_2^b, \quad \text{ with } a = b = 1$$

Then, the inverse demand function for item 1 is

$$p (x_1) = \left(\frac{a}{a+b}\right) \frac{m}{x_1}$$

Example 25 *Suppose that the consumer has a quasi-linear preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = \ln x_1 + x_2$$

Then,

$$x_1(p_1, p_2, m) = \frac{p_2}{p_1} \implies p(x_1) = \frac{p_2}{x_1}$$

Remark 24 *If $p_2 = 1$, the tangency condition implies that*

$$p_1 = |MRS|$$

This means that the price p_1 measures how much the consumer is willing to give up of item 2 in order to get a little more of item 1.

Remark 25 *If item 2 is money, this means that p_1* ***measures the marginal willingness to pay*** *of our consumer. In other words, at each quantity x_1, the inverse demand function measures how many British pounds the consumer is willing to give up to a little more of item 1*

7 Market Demand

***Assumption 3** Suppose that in the market there are only two consumers, Alice and Bob. Denote Alice's demand of item $i = 1, 2$ by $x_A(p_i)$ and denote Bob's demand of item $i = 1, 2$ by $x_B(p_i)$.*

***Definition 34** The **market demand** for item $i = 1, 2$, also called the aggregate demand for item $i = 1, 2$, denoted by $X(p_i)$, is the (horizontal) sum of the individual demands over all consumers in the market*

$$X(p_i) = x_A(p_i) + x_B(p_i)$$

***Definition 35** The **inverse market demand** for item $i = 1, 2$, also called the aggregate inverse demand for item $i = 1, 2$, denoted by $P(X_1)$, measures what the market price for item i would have to be for X_i units of it to be demanded*

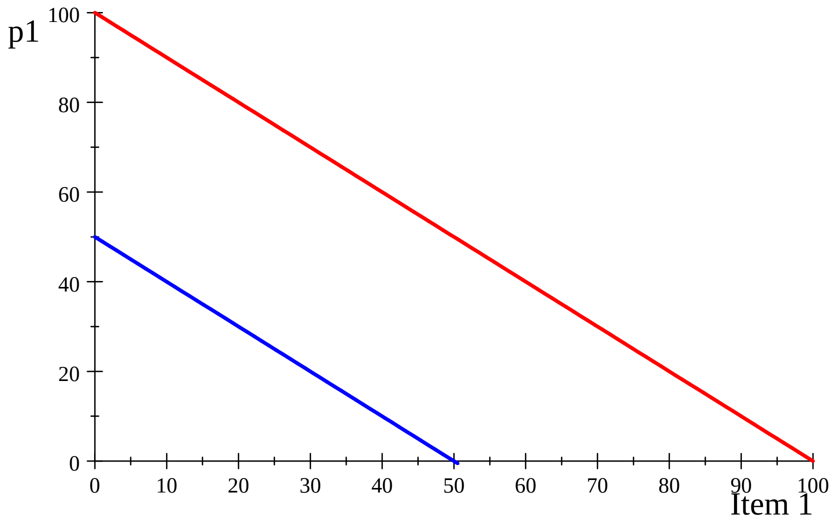
***Remark 26** We seen that when $p_2 = 1$ the tangency condition implies that*

$$p_1 = |MRS|$$

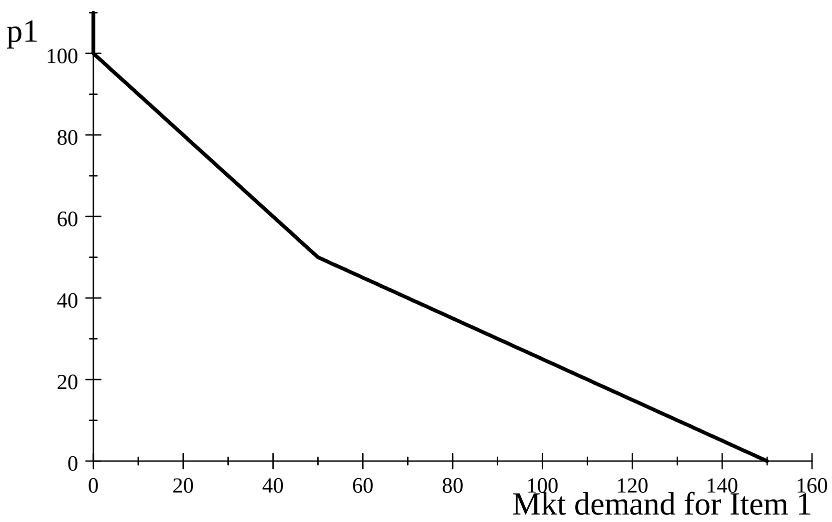
Thus, $P(X_i)$ measures the marginal willingness to pay of every consumer who is purchasing the item i.

Example 26 *Suppose that* $x_A(p_1) = 100 - p_1$ *and* $x_B(p_1) = 50 - p_1$. *Then, the market demand for item 1 is*

$$X(p_1) = \begin{cases} 150 - 2p_1 & \textit{if} \quad p_1 \leq 50 \\ 100 - p_1 & \textit{if} \quad 100 \geq p_1 > 50 \\ 0 & \textit{if} \quad p_1 > 100 \end{cases}$$



Individual demand curves



Mkt demand curve

Definition 36 *The price elasticity of demand for item i is the percentage change in the amount demanded for item i as its own price p_i changes by 1 percent.*

$$\epsilon_{X_i,p_i} = \underbrace{\left(\frac{dX(p_i)}{dp_i}\right)}_{\text{Slope of } X(p_i)} \frac{p_i}{X(p_i)}$$

Example 27 *Let $X(p_1) = 100 - p_1$. Then,*

$$\epsilon_{X_1,p_1} = (-1)\frac{p_1}{100 - p_1}$$

At point $(50, 50)$, the price elasticity of demand for item 1 is equal to $\epsilon_{X_1,p_1} = -1$.

Example 28 *Let $X(p_1) = a - bp_1$ with $a, b > 0$. Then,*

$$|\epsilon_{X_1,p_1}| = \left|(-b)\frac{p_1}{a - bp_1}\right| = \begin{cases} \infty & \textit{if} \quad p_1 = \frac{a}{b} \\ > 1 \textit{ (Elastic)} & \textit{if} \quad \frac{a}{b} > p_1 > \frac{a}{2b} \\ = 1 \textit{ (Unit Elastic)} & \textit{if} \quad p_1 = \frac{a}{2b} \\ < 1 \textit{ (Inelastic)} & \textit{if} \quad \frac{a}{2b} > p_1 > 0 \\ 0 & \textit{if} \quad p_1 = 0 \end{cases}$$

8 Slutsky Decomposition

Notation 1 *Suppose that the utility function $u(\cdot)$ represents the consumer's preference $\succsim$. Suppose that p_i is the price of item $i = 1, 2$. Suppose that the consumer's income is m. Then, the individual demand for item 1 is*

$$x_1(p_1, m) \equiv x_1(p_1, p_2, m)$$

I have dropped p_2 since from now onwards I will keep it constant. Then, the individual demand for item 1 is a function of two variables, namely, p_1 and m.

Notation 2 *Initially, I suppose that the price of item 1 is p_1 (**old price**). However, there is a price change in the market for this item. The new price will be denoted by p_1' (**new price**). The change in price will be denoted by*

$$\Delta p_1 = (p_1' - p_1),$$

*the new individual demand for the item 1 at the new price p_1' and **old income** m will be denoted by*

$$x_1(p_1', m),$$

*and the **total change in the demand** for item 1 will be*

$$\Delta x_1 = x_1(p_1', m) - x_1(p_1, m)$$

Definition 38 *The change in demand $\triangle x_1$ can be decomposed in two part, one part is called* ***substitution effect****, $\triangle x_1^s$, whereas the other part is called* ***income effect****, $\triangle x_1^n$*

Definition 39 *Suppose that before the price change of item 1, the optimal consumption bundle at (p_1, p_2, m) is (x_1, x_2). Now suppose that the price of item 1 change from p_1 to p_1'. The amount of money, denoted by m^{hyp}, that would make the bundle (x_1, x_2) just affordable at new price system (p_1', p_2) would be*

$$m^{hyp} = p_1' x_1 + p_2 x_2.$$

The old income m is equal to

$$m = p_1 x_1 + p_2 x_2.$$

The income of the consumer needs to be adjusted by

$$\Delta m = \left(m^{hyp} - m\right) = \Delta p_1 x_1 \text{ where } \Delta p_1 = (p_1' - p_1)\,.$$

Definition 40 *The **substitution effect**, Δx_1^s, is the change in the demand for item 1 when the price of item 1 change to p_1' and, at the same time, income changes to m^{hyp}:*

$$\Delta x_1^s = x_1\left(p_1', m^{hyp}\right) - x_1\left(p_1, m\right)$$

Definition 41 *The **income effect**, Δx_1^n, is the change in the demand for item 1 when change income from m^{hyp} to m, holding the price of item 1 fixed at p_1':*

$$\Delta x_1^n = x_1\left(p_1', m\right) - x_1\left(p_1', m^{hyp}\right)$$

Definition 42 *The **total change** in demand is*

$$\Delta x_1 = \Delta x_1^s + \Delta x_1^n \text{ (Slutsky identity)}$$

Example 29 *Suppose that the consumer has a Cobb-Douglas preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = x_1^a x_2^b, \quad \text{with } a = b = 1$$

Then, the individual demand functions are

$$x_1(p_1, p_2, m) = \frac{m}{2p_1} \quad \text{and} \quad x_2(p_1, p_2, m) = \frac{m}{2p_2}$$

Initially, let $p_1 = p_2 = 1$ and $m = 18$. Suppose that the price of item 1 change to $p_1' = 2.25$
At (p_1, p_2, m), the optimal consumption bundle is

$$x_1^{old} = x_2^{old} = 9.$$

At (p_1', p_2, m), the optimal consumption bundle is

$$x_1^{new} = 4,\ x_2^{new} = 9.$$

The income of the consumer needs to be adjusted by

$$\Delta m = \Delta p_1 x_1^{old} = 1.25 x_1^{old} = 11.25$$

so that

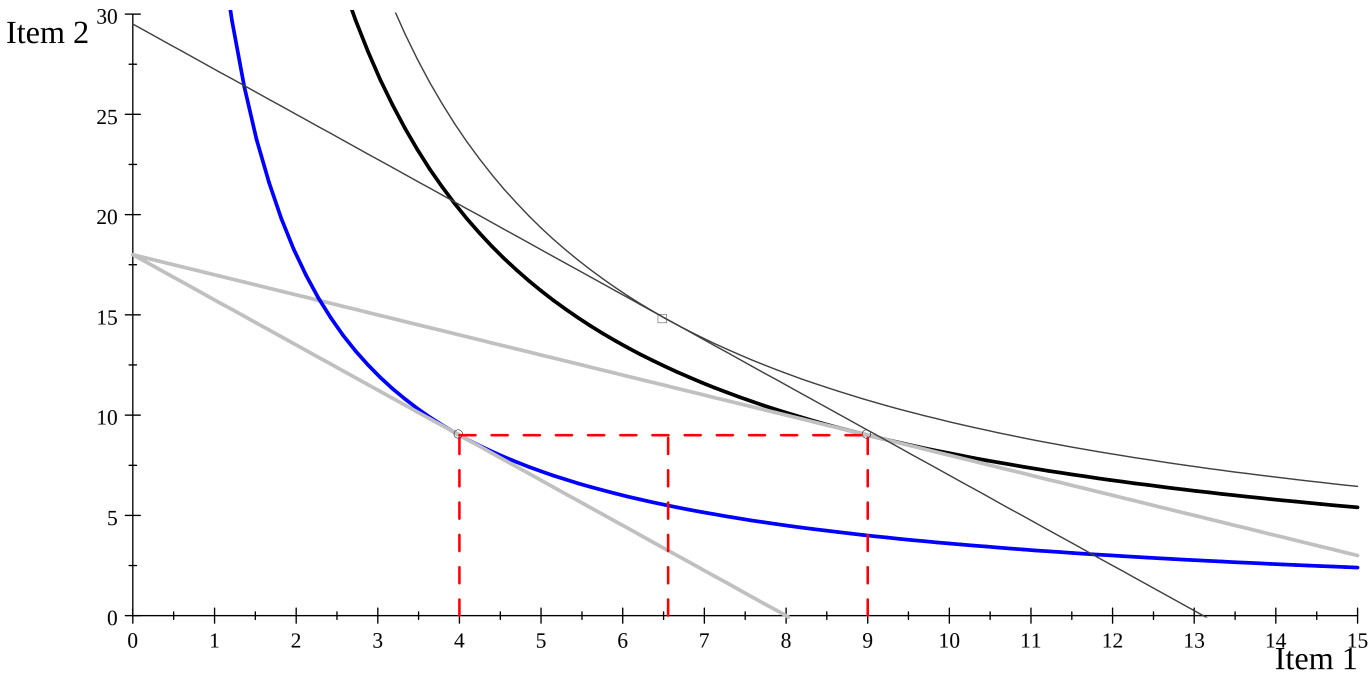
$$m^{hyp} = 18 + 11.25 = 29.25$$

At (p_1', p_2, m')*, the optimal consumption bundle is*

$$y_1^{hyp} = 6.555\,6,\ y_2^{hyp} = 14.75.$$

Thus:

$$\Delta x_1^s = -2.4444, \qquad \Delta x_1^n = -2.555\,6 \quad \textit{and} \quad \Delta x_1 = -5$$



$$k = 36, 81, 96.695$$

Remark 29 (Normal good) *The substitution effect moves opposite to the price movement, we say it is always negative:*

- If the price goes down, the quantity demanded via the substitution effect is non-negative, $\Delta x_1^s \geq 0$.
- If the price goes up, the quantity demanded via the substitution effect is non-positive, $\Delta x_1^s \leq 0$.

The income effect also moves opposite to the price movement, we say it is negative:

- If the price goes down, the quantity demanded via the income effect is non-negative, $\Delta x_1^n \geq 0$.
- If the price goes up, the quantity demanded via the income effect is non-positive, $\Delta x_1^n \leq 0$.

The total change in demand due to a price change for a normal good:

- is always non-negative if the price goes down, $\Delta x_1 \geq 0$.
- is always non-positive if the price goes up, $\Delta x_1 \leq 0$.

Remark 30 (Inferior/Giffen good) *For an inferior good, the substitution effect is always negative. For an inferior good, the income effect moves in the same direction of the price movement:*

- If the price goes down, the quantity demanded via the income effect is non-positive, $\Delta x_1^n \leq 0$.
- If the price goes up, the quantity demanded via the income effect is non-negative, $\Delta x_1^n \geq 0$.

The total change in demand due to a price change for an inferior good is ambiguous:

- The good is a Giffen good if the income effect Δx_1^n outweighs the substitution effect Δx_1^s, that is:
  - $\Delta x_1 > 0$ if the price goes up.
  - $\Delta x_1 < 0$ if the price goes down.

9 Hicks Decomposition

Method 4 *Let* $\left(x_1^{old}, x_2^{old}\right) = \left(x_1\left(p_1, m\right), x_2\left(p_1, m\right)\right)$ *be the utility-maximizing bundle at* (p_1, p_2, m)*. The* ***Hicks (compensating variation) method*** *is based on a hypothetical budget line that is tangent to the original indifference curve at the level* $u\left(x_1^{old}, x_2^{old}\right) = k$ *but with a slope based on the new prices. The income of this hypothetical budget line is such that*

$$m^H = p_1' \cdot x_1\left(p_1', m^H\right) + p_2 \cdot x_2\left(p_1', m^H\right) \quad \textit{where} \quad u\left(x_1^{old}, x_2^{old}\right) = u\left(x_1\left(p_1', m^H\right), x_2\left(p_1', m^H\right)\right).$$

Definition 43 *The* ***Hicks substitution effect*** *is thus*

$$\Delta x_1^s = x_1\left(p_1', m^H\right) - x_1\left(p_1, m\right).$$

Definition 44 *The* ***Hicks income effect*** *is thus*

$$\Delta x_1^n = x_1\left(p_1', m\right) - x_1\left(p_1', m^H\right).$$

Definition 45 *The* ***British pounds value of income effect*** *is given by*

$$m - m^H,$$

which measures the consumer's loss or gain in British pounds of the price change. In other words, it is the (Hicks) change in utility measured in monetary terms.

Example 30 *Suppose that the consumer has a Cobb-Douglas preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = x_1^a x_2^b, \quad \textit{with } a = b = 1$$

Then, the individual demand functions are

$$x_1(p_1, p_2, m) = \frac{m}{2p_1} \quad \textit{and} \quad x_2(p_1, p_2, m) = \frac{m}{2p_2}$$

Initially, let $p_1 = p_2 = 1$ *and* $m = 18$*. Suppose that the price of item 1 change to* $p_1' = 2.25$
At (p_1, p_2, m)*, the optimal consumption bundle is*

$$x_1^{old} = x_2^{old} = 9.$$

At (p_1', p_2, m)*, the optimal consumption bundle is*

$$x_1(p_1', m) = x_1^{new} = 4, \; x_2^{new} = x_2(p_1', m) = 9.$$

To find $\left(x_1\left(p_1', m^H\right), x_2\left(p_1', m^H\right)\right)$ *we use the fact that*

$$81 = u\left(x_1^{old}, x_2^{old}\right) = u\left(x_1\left(p_1', m^H\right), x_2\left(p_1', m^H\right)\right) = x_1\left(p_1', m^H\right) \cdot x_2\left(p_1', m^H\right)$$

and the fact that the tangency condition requires

$$MRS = -\frac{x_2\left(p_1', m^H\right)}{x_1\left(p_1', m^H\right)} = -\frac{p_1'}{p_2} = -2.25.$$

Putting these two equations together gives

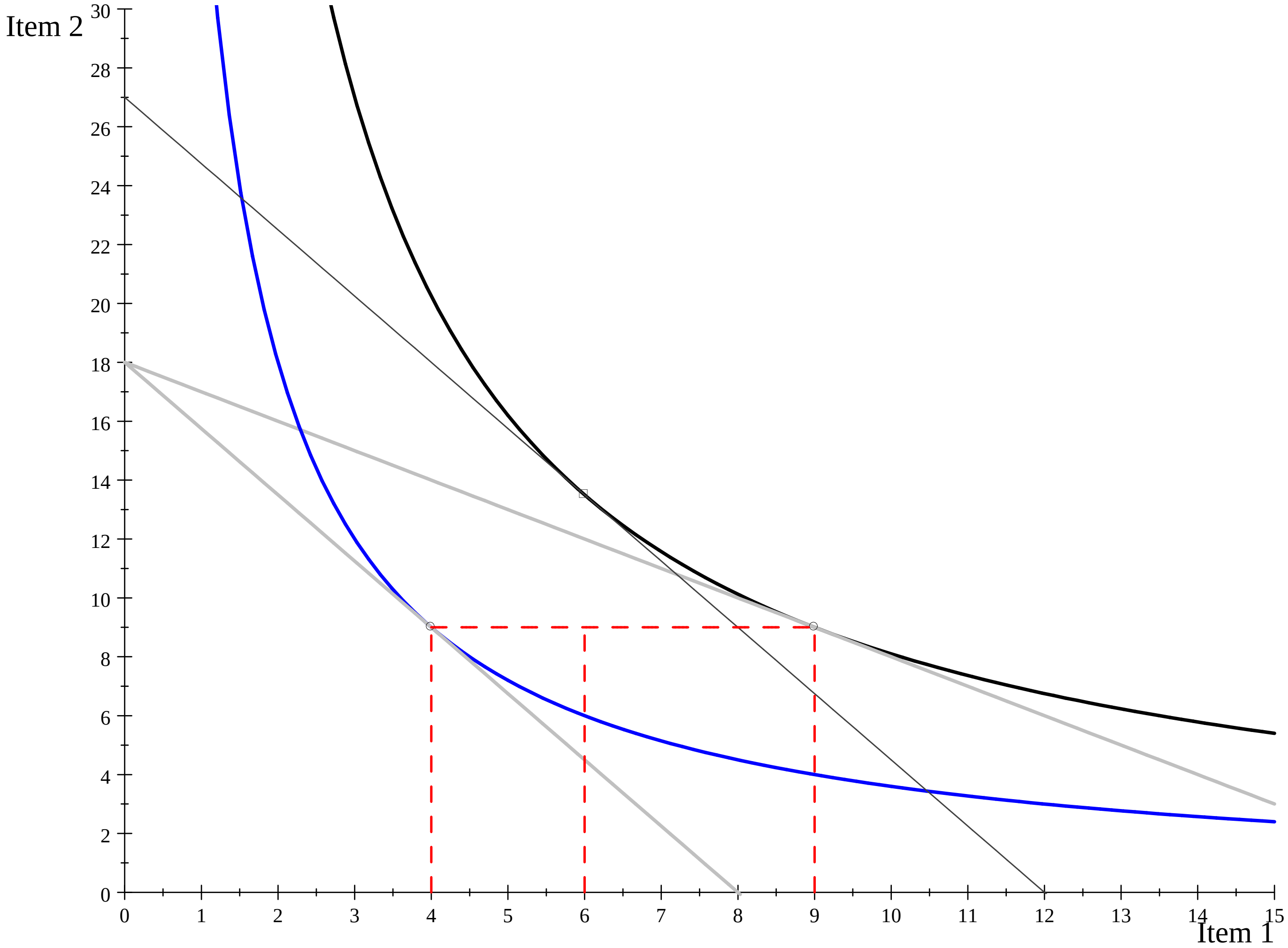
$$x_1\left(p_1', m^H\right) = 6 \quad \textit{and} \quad x_2\left(p_1', m^H\right) = 13.5.$$

The new income is thus

$$m^H = 2.25 \times 6 + 1 \times 13.5 = 27$$

Thus:

$$\Delta x_1^s = -3, \quad \Delta x_1^n = -2 \quad \textit{and} \quad \Delta x_1 = -5$$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Item 1

$$k = 36, 81$$

10 Kaldor Decomposition

Method 5 *Let* $(x_1^{new}, x_2^{new}) = (x_1(p_1', m), x_2(p_1', m))$ *be the utility-maximizing bundle at* (p_1', p_2, m)*. The* ***Kaldor (equivalent variation) method*** *is based on a hypothetical budget line that is tangent to the new indifference curve at the level* $u(x_1^{new}, x_2^{new}) = k$ *but with a slope based on the old prices. The income of this hypothetical budget line is such that*

$$m^K = p_1 x_1\left(p_1, m^K\right) + p_2 x_1\left(p_1, m^K\right) \qquad \textit{where} \qquad u(x_1^{new}, x_2^{new}) = u\left(x_1\left(p_1, m^K\right), x_2\left(p_1, m^K\right)\right).$$

Definition 46 *The* ***Kaldor substitution effect*** *is thus*

$$\Delta x_1^s = x_1(p_1', m) - x_1\left(p_1, m^K\right).$$

Definition 47 *The* ***Kaldor income effect*** *is thus*

$$\Delta x_1^n = x_1\left(p_1, m^K\right) - x_1(p_1, m).$$

Definition 48 *The* ***British pounds value of income effect*** *is given by*

$$m^K - m,$$

which measures the consumer's loss or gain in British pounds of the price change. In other words, it is the (Kaldor) change in utility measured in monetary terms.

Example 31 *Suppose that the consumer has a Cobb-Douglas preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = x_1^a x_2^b, \quad \text{with } a = b = 1$$

Then, the individual demand functions are

$$x_1(p_1, p_2, m) = \frac{m}{2p_1} \quad \text{and} \quad x_2(p_1, p_2, m) = \frac{m}{2p_2}$$

Initially, let $p_1 = p_2 = 1$ *and* $m = 18$*. Suppose that the price of item 1 change to* $p_1' = 2.25$
At (p_1, p_2, m)*, the optimal consumption bundle is*

$$x_1^{old} = x_2^{old} = 9.$$

At (p_1', p_2, m)*, the optimal consumption bundle is*

$$x_1^{new} = 4, \; x_2^{new} = 9.$$

To find $\left(x_1\left(p_1, m^K\right), x_2\left(p_1, m^K\right)\right)$ *we use the fact that*

$$u\left(x_1^{new}, x_2^{new}\right) = 36 = u\left(x_1\left(p_1, m^K\right), x_2\left(p_1, m^K\right)\right) = x_1\left(p_1, m^K\right) \cdot x_2\left(p_1, m^K\right)$$

and the fact that the tangency condition requires

$$MRS = -\frac{x_2\left(p_1, m^K\right)}{x_1\left(p_1, m^K\right)} = -\frac{p_1}{p_2} = -1.$$

Putting these two equations together gives

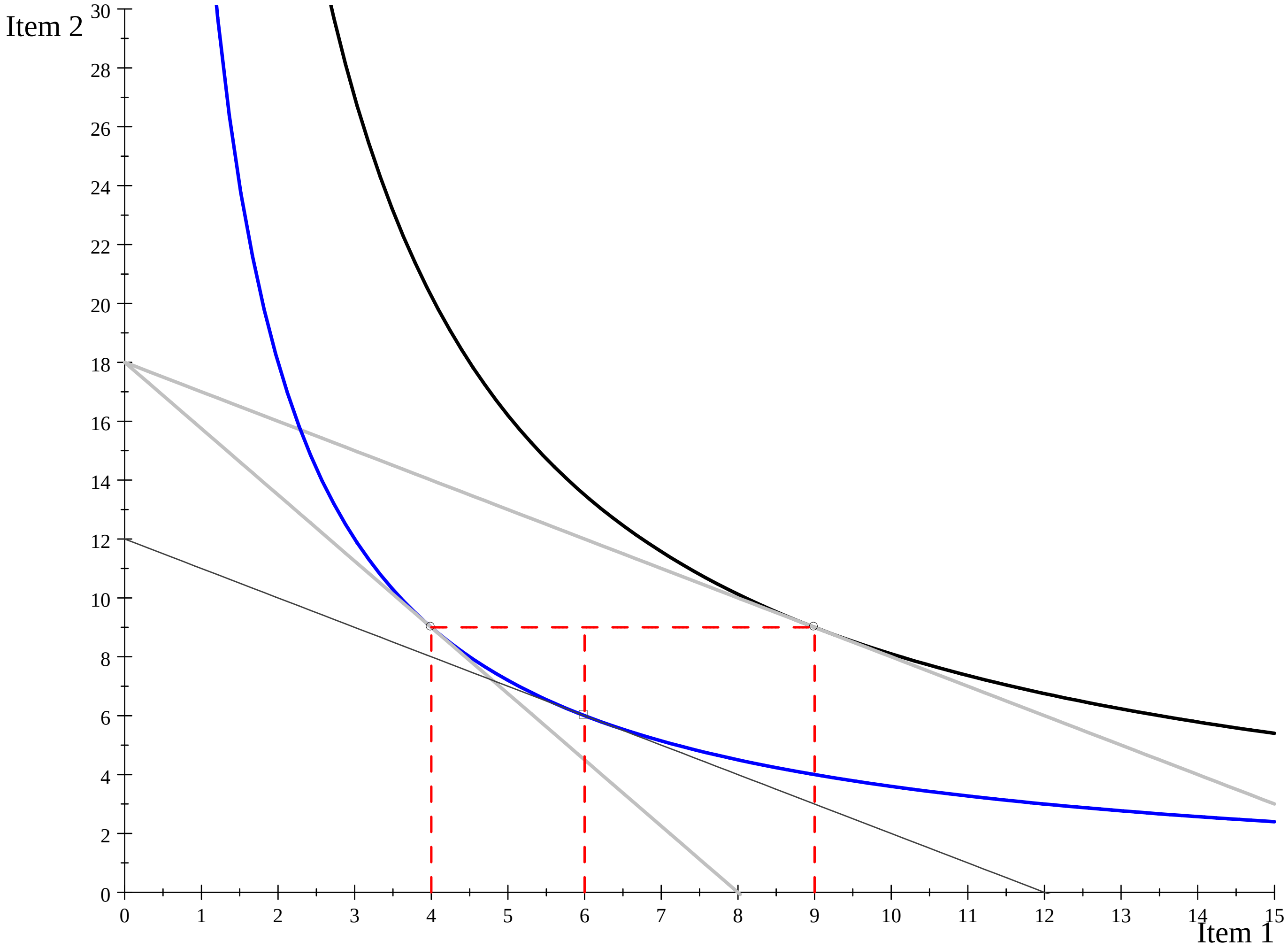
$$x_1\left(p_1, m^K\right) = 6 \quad \text{and} \quad x_2\left(p_1, m^K\right) = 6.$$

The new income is thus

$$m^K = 1 \times 6 + 1 \times 6 = 12$$

Thus:

$$\Delta x_1^s = -2, \quad \Delta x_1^n = -3 \quad \text{and} \quad \Delta x_1 = -5$$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Item 1

$$k = 36, 81$$

Definition 49 Compensating Variation (CV) *- CV is how much money we would have to give to (or take away from) the consumer to get him back to the same level of utility that he had before prices changed. So to calculated CV, you try to get the consumer to the initial utility level at the new prices by changing income. That is,*

$$m^H - m.$$

Definition 50 Equivalent Variation (EV) *- EV is how much money the consumer would be willing to give up (or be paid) to prevent prices from changing. Thus to get EV, we get the consumer to the different utility level under old prices by changing income. That is,*

$$m - m^K.$$

Example 32 *Suppose that the consumer has a Cobb-Douglas preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = x_1^a x_2^b, \quad \text{with } a = b = 1$$

Then, the individual demand functions are

$$x_1(p_1, p_2, m) = \frac{m}{2p_1} \quad \text{and} \quad x_2(p_1, p_2, m) = \frac{m}{2p_2}$$

Initially, suppose that $p_1 = p_2 = 1$ *and* $m = 18$*. Then, the price of item 1 increases to* $p_1' = 2.25$*.*
The optimal consumption bundles are

$$x_1^{old} = x_2^{old} = 9 \quad \text{and} \quad x_1^{new} = 4,\ x_2^{new} = 9.$$

We can find m^H *by solving the following equation:*

$$\left(\frac{m^H}{2 \times 2.25}\right)\left(\frac{m^H}{2 \times 1}\right) = u(9, 9) \implies m^H = 27\text{: } CV = m^H - m = 9$$

We can find m^K *by solving the following equation:*

$$\left(\frac{m^K}{2 \times 1}\right)\left(\frac{m^K}{2 \times 1}\right) = u(4, 9) \implies m^H = 12\text{: } EV = m - m^K = 6$$

Example 33 *Suppose that the consumer has a Cobb-Douglas preference, that is, his preference is represented by the utility function* $u(x_1, x_2) = x_1^a x_2^b$, *with* $a = b = 1$. *Then, the individual demand functions are*

$$x_1(p_1, p_2, m) = \frac{m}{2p_1} \quad \text{and} \quad x_2(p_1, p_2, m) = \frac{m}{2p_2}$$

Initially, suppose that $p_1 = 2.25$, $p_2 = 1$ *and* $m = 18$. *Then, the price of item 1 decreases to* $p_1' = 1$. *The optimal consumption bundles are*

$$x_1^{old} = 4,\ x_2^{old} = 9, \quad \text{and} \quad x_1^{new} = x_2^{new} = 9$$

We can find m^H *by solving the following equation:*

$$\left(\frac{m^H}{2 \times 1}\right)\left(\frac{m^H}{2 \times 1}\right) = u(4, 9) \implies m^H = 12\text{: } CV = m^H - m = -6$$

We can find m^K *by solving the following equation:*

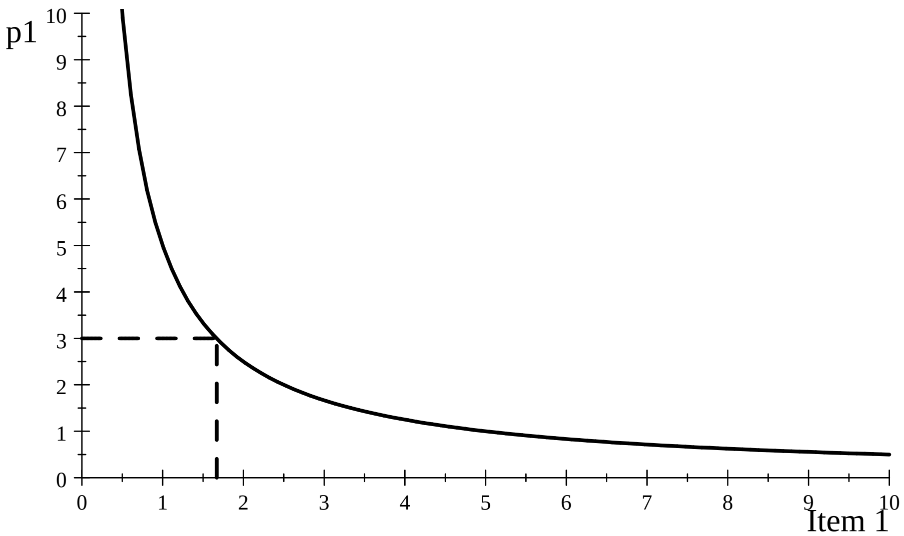
$$\left(\frac{m^K}{2 \times 2.25}\right)\left(\frac{m^K}{2 \times 1}\right) = u(9, 9) \implies m^K = 27\text{: } EV = m - m^K = -9$$

Remark 31 *The absurdity of CVs and EVs is due to the ambiguity of income effects for some utility functions. This absurdity disappear if a consumer has a quasilinear preference (no income effect!).*

11 Consumer's surplus

Definition 51 *The consumer's surplus for item $i = 1, 2$ is the aggregate amount, over all units consumed of item i, of the consumer's willingness to pay for the additional units, minus the amount actually paid.*

Example 34 *Suppose that $p(x_1) = \frac{5}{x_1}$. Suppose that $p_1 = 3$. Then, $x_1 = 1.67$. The consumer's surplus is the area under the inverse demand curve and above the horizontal line at $p_1 = 3$.*



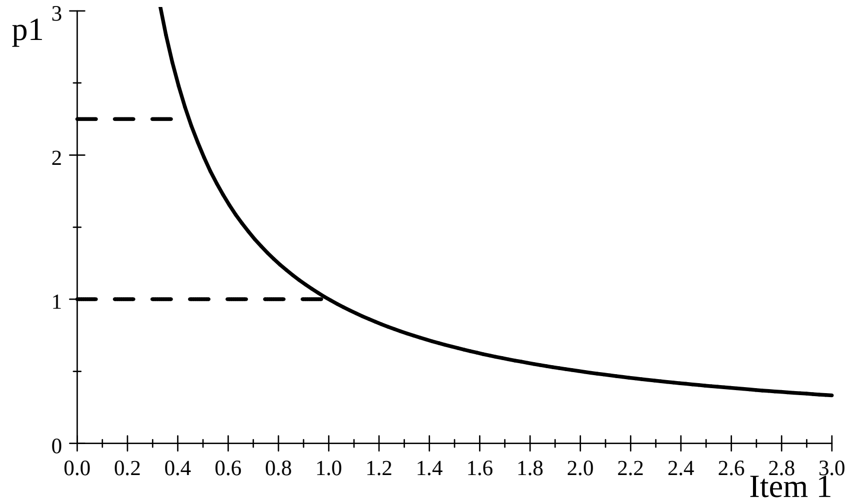
Example 35 *Suppose that the consumer has a quasi-linear preference represented by the utility function* $u(x_1, x_2) = \ln x_1 + x_2$*. Then, the individual demand functions are*

$$x_1(p_1, p_2, m) = \frac{p_2}{p_1} \quad \text{and} \quad x_2(p_1, p_2, m) = \frac{m}{p_2} - 1$$

Initially, suppose that $p_1 = p_2 = 1$ *and* $m = 18$*. Then, the price of item 1 increases to* $p_1' = 2.25$*. Thus*

$$x_1^{old} = 1 \text{ and } x_2^{old} = 17;\ x_1^{new} = \frac{4}{9} \text{ and } x_2^{new} = 17.$$

The change in the consumer's surplus is



$$\Delta CS = \frac{5}{9} + \int_{4/9}^{1} \frac{1}{x_1} dx_1 - \frac{5}{9} = \ln 1 - \ln\left(\frac{4}{9}\right) = 0.81$$

12 Intertemporal Choice

Assumption 4 *Let r denote the mkt interest rate. Suppose that there only two dates, $t = 1, 2$. Consumer's income at date t is denoted by m_t. After $t = 2$ the consumer dies.*

Remark 32 *Suppose that at $t = 1$ (today) you have £1. What is its future value (**FV**) at $t = 2$ (tomorrow)? Answer: $£1 \times (1 + r)$*

Remark 33 *Suppose that we are in $t = 1$ (today). Suppose that at $t = 2$ (tomorrow) you will have £1. What is its present value (**PV**) at $t = 1$? Answer: $\frac{£1}{1+r}$*

Remark 34 *Suppose that at $t = 1$ the consumer's income is m_1 and at $t = 2$ the consumer's income will be m_2. The value of the consumer's income stream is*

$$PV = m_1 + \frac{m_2}{1 + r} \quad \textit{and} \quad FV = m_1 (1 + r) + m_2$$

Assumption 5 *Let x_t denote the consumption in $t = 1, 2$ that is measured in monetary terms. The consumer is able to borrow/lend at the mkt interest rate r. Let (m_1, m_2) be the endowment (income stream) of the consumer.*

Remark 35 *The consumer is a lender if*

$$m_1 - x_1 > 0$$

Then, the consumer's consumption in $t = 2$ is

$$\begin{aligned} x_2 &= m_2 + (m_1 - x_1) + r(m_1 - x_1) \\ &= m_2 + (1 + r)(m_1 - x_1) \end{aligned}$$

Remark 36 *The consumer is a borrower if*

$$m_1 - x_1 < 0$$

Then, the consumer's consumption in $t = 2$ is

$$\begin{aligned} x_2 &= m_2 - (x_1 - m_1) - r(x_1 - m_1) \\ &= m_2 + (1 + r)(m_1 - x_1) \end{aligned}$$

Definition 52 *The **intertemporal budget equation** is equal to*

$$x_1(1+r) + x_2 = m_1(1+r) + m_2 \qquad \textit{(FV),}$$

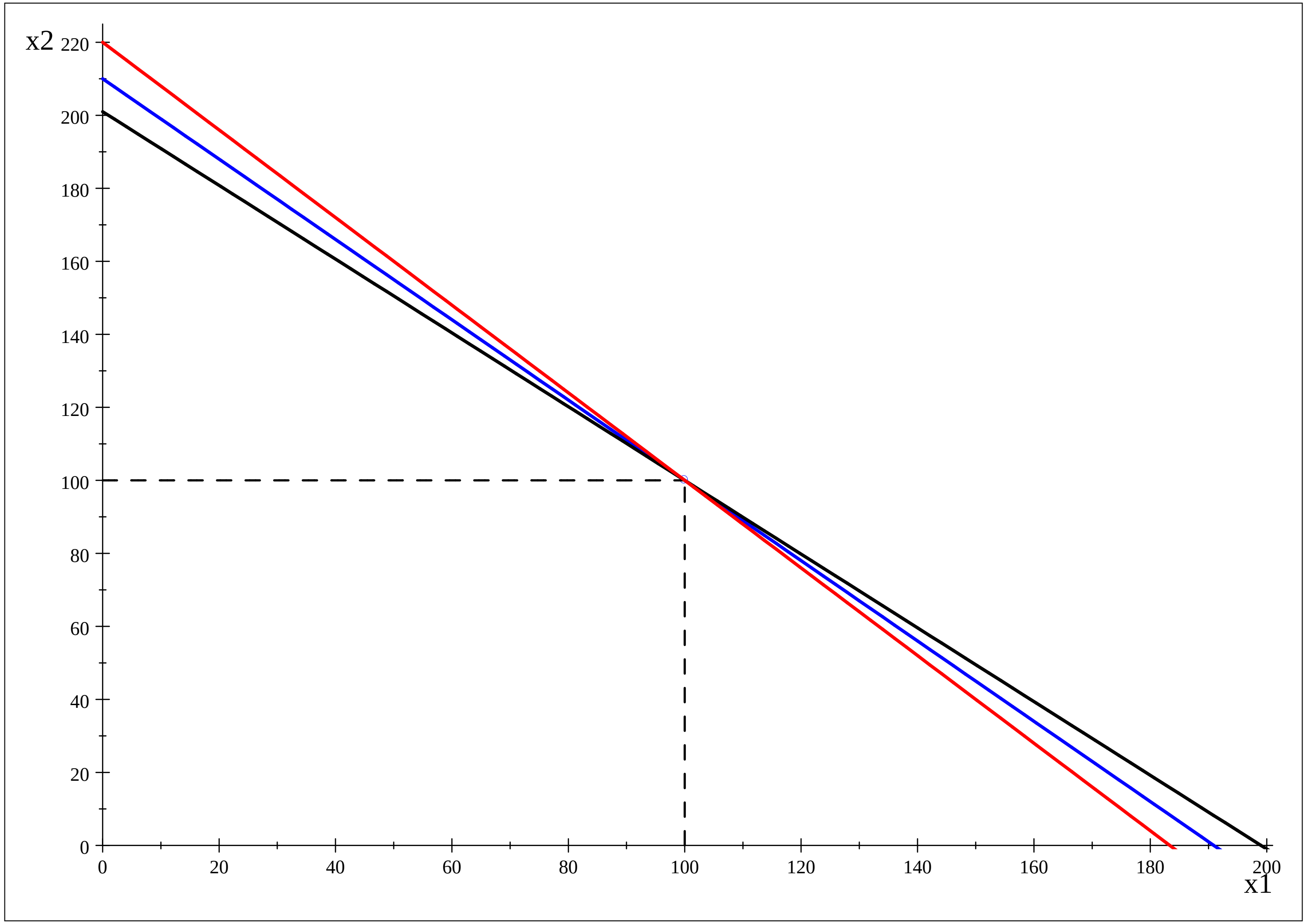
or equivalently,

$$x_1 + \frac{x_2}{1+r} = m_1 + \frac{m_2}{1+r} \qquad \textit{(PV)}$$

Its slope is

$$-(1+r)$$

Example 36 *Suppose that* $m_t = 100$ *for* $t = 1, 2$. *Suppose that* $r = 0.01$ *(black line)*, 0.1 *(blue line) and* 0.2 *(red line).*



Remark 37 *In this setting, perfect-substitute preferences do not make much sense : the consumer does not care whether he consumes today or tomorrow!*

Perfect-complement preferences do not make much sense either. The reason is that the consumer would like to consume the item today and tomorrow in a fixed proportion. Thus, he is unwilling to substitute consumption from period to another, no matter how high (or low) is the mkt interest rate

Well-behaved preferences make sense here.

Problem 3 *Suppose that the utility function* $u(\cdot)$ *represents the consumer's preference* $\succsim$ *over consumption bundles* (x_1, x_2)*, where* x_t *is the consumption in period* $t = 1, 2$*. Suppose that the consumer's income stream is* (m_1, m_2)*. Suppose that* r *is the mkt interest rate.*

The consumer optimal's consumption bundle (x_1^*, x_2^*) *is the solution to this* ***constraint maximization problem****:*

$$\underset{x_1, x_2}{\text{Maximize}} \quad u(x_1, x_2)$$

subject to (the intertemporal budget constraint)

$$x_1 + \frac{x_2}{1+r} = m_1 + \frac{m_2}{1+r}$$

Example 37 *Suppose that the consumer has a Cobb-Douglas preference, that is, his preference is represented by the utility function*

$$u(x_1, x_2) = x_1 x_2, \quad \textit{with } a = b = 1$$

Suppose that $r = 0.1$*, that* $m_t = 100$ *for* $t = 1, 2$*.*

$$MRS = -\frac{x_2}{x_1} = -(1 + r) = -1.1.$$

This gives

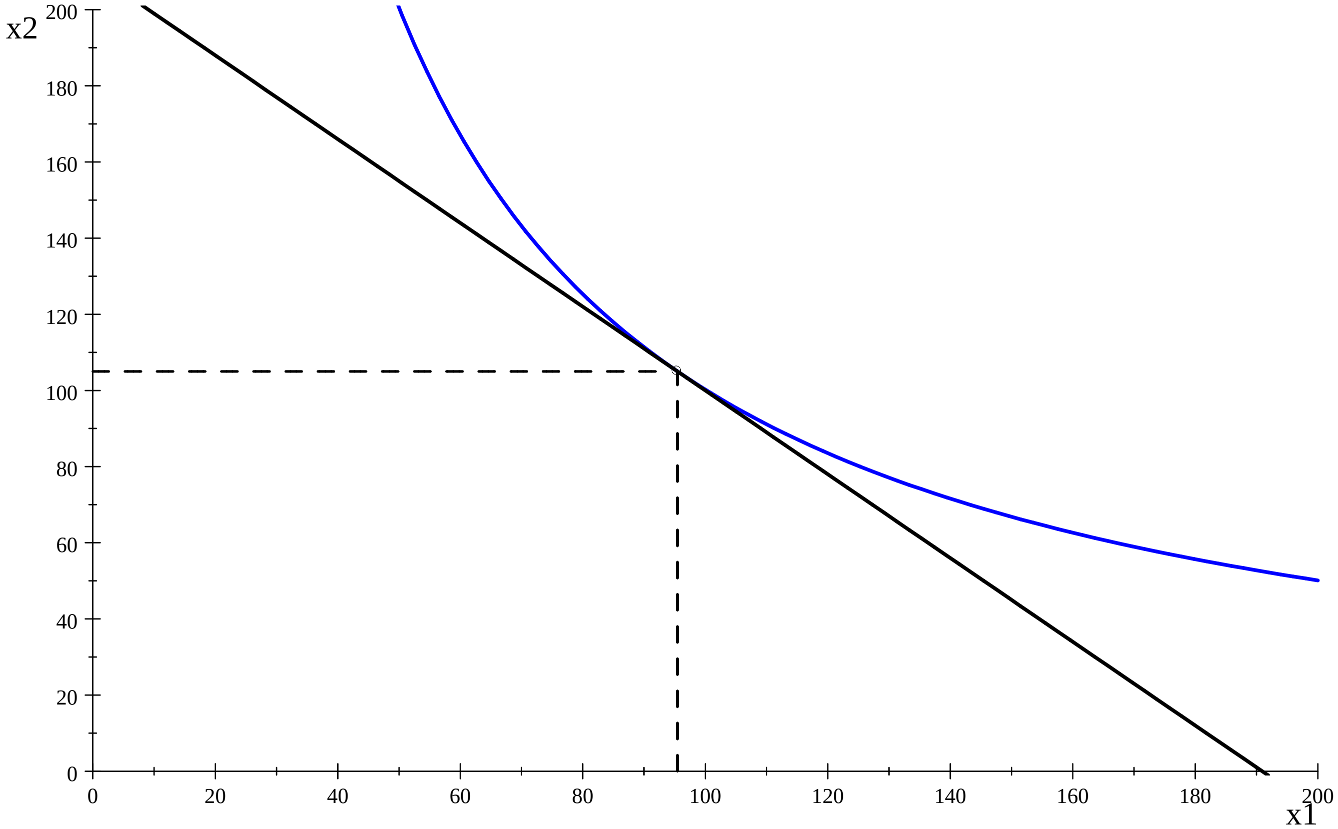
$$x_2 = 1.1x_1 \quad \textit{or} \quad x_1 = \frac{x_2}{1.1}.$$

Plugging $x_2 = 1.1x_1$ *into the intermporal budget equation gives*

$$x_1 = 95.455$$

Plugging $x_1 = \frac{x_2}{1.1}$ *into the intermporal budget equation gives*

$$x_2 = 105$$


x2
200
180
160
140
120
100
80
60
40
20
0
0
20
40
60
80
100
120
140
160
180
200
x1

13 Game Theory

Definition 53 *A strategic game consists of (a) a set of* ***players****, (b) for each player, a set of* ***actions****, and (c) for each player,* ***preferences*** *of the set of action profiles*

Remark 38 *A very wide range of situations can be modeled as strategic games. For example, the players can be firms, the action prices, and the preferences a reflection of the firms' profits. Or the players may be candidates for political office, the action campaign expenditures, and the preferences a reflection of the candidate probability of winning.*

13.1 Prisoner's Dilemma

Player 2 (column player)

Player 1 (row player)		*Quiet*	*Fink*
	Quiet	$(2, 2)$	$(0, 3)$
	Fink	$(3, 0)$	$(1, 1)$

Remark 39 *The Prisoner's Dilemma models a situation in which there are gains from cooperation (each player prefers that both players choose Quiet than the both chose Fink) but each player has an incentive to "free ride" (choose Fink) whatever the other player does. The game is important not because we are interested in understanding the incentives fro prisoners to confess, but because many other situations have similar structures.*

Player 2 (column player)

Player 1 (row player)		*Work hard*	*Goof off*
	Work hard	$(2, 2)$	$(0, 3)$
	Goof off	$(3, 0)$	$(1, 1)$

13.2 Battle of Sex (BoS)

Remark 40 *In the Prisoner's Dilemma the main issue is whether the players will cooperate (choose Quiet). In the battle of sex the players agree that it is better to cooperate that not to cooperate, but they disagree about the best outcome. It is a coordination game.*

		Player 2 (column player)	
		Bach	*Stravinsky*
Player 1 (row player)	*Bach*	$(2, 1)$	$(0, 0)$
	Stravinsky	$(0, 0)$	$(1, 2)$

Remark 41 *BoS models a wide variety of situations. Consider two merging firms that currently use different computer technologies. As two divisions of a single firm they will both be better off if they both use the same technology; each firm prefers that the common technology be the one it used in the past.*

13.3 Matching Pennies

Remark 42 *Aspect of both conflict and cooperation are present in the Prisoner's Dilemma and BoS. The matching pennies game is pure conflictual.*

Player 2 (column player)

		Head	*Tail*
Player 1 (row player)	*Head*	$(1, -1)$	$(-1, 1)$
	Tail	$(-1, 1)$	$(1, -1)$

Remark 43 *In this game the players' interests are diametrically opposed: player 1 wants to take the same action as the other player, whereas player 2 wants to take the opposite action.*

Remark 44 *This game may, for example, model the choices of appearances for new products by an established producer and a new firm in a market of fixed size. Suppose that each firm can choose one of two different appearances for the product. The established producer prefers the newcomer's product to look different from its own (so that its customers will not be tempted to buy the newcomer's product), whereas the newcomer prefers that the products look like.*

13.4 Nash Equilibrium

Definition 54 *An action profile* (a_1^*, a_2^*) *is a* ***Nash equilibrium*** *if, for every player* $i = 1, 2$*, it holds that*

$$u_1(a_1^*, a_2^*) \geq u_1(a_1, a_2^*) \text{ for every other action } a_1 \text{ of player 1,}$$

and

$$u_2(a_1^*, a_2^*) \geq u_2(a_1^*, a_2) \text{ for every other action } a_2 \text{ of player 2.}$$

13.5 Nash Equilibrium - Prisoner's Dilemma

		Player 2 (column player)	
		Quiet	*Fink*
Player 1 (row player)	*Quiet*	$(2, 2)$	$(0, \underline{3})$
	Fink	$(\underline{3}, 0)$	$(\underline{1}, \underline{1})$

Remark 45 *The Nash equilibrium action profile is* $(Fink, Fink)$*, that is, player 1 plays Fink, and player 2 plays Fink. The Nash equilibrium outcome is* $(1, 1)$*, that is, player 1's utility is 1 while player 2's utility is 1.*

13.6 Nash Equilibrium - BoS

		Player 2 (column player)	
		Bach	*Stravinsky*
Player 1 (row player)	*Bach*	$(\underline{2}, \underline{1})$	$(0, 0)$
	Stravinsky	$(0, 0)$	$(\underline{1}, \underline{2})$

Remark 46 *There are two Nash equilibrium action profiles. One Nash equilibrium action profile is* $(Bach, Bach)$*. The corresponding Nash equilibrium outcome is* $(2, 1)$*, player 1's utility is 2 while player 2's utility is 1. The other Nash equilibrium action profile is* $(Stravinsky, Stravinsky)$*. The corresponding Nash equilibrium outcome is* $(1, 1)$*, player 1's utility is 1 while player 2's utility is 2.*

13.7 Nash Equilibrium - Matching pennies

		Player 2 (column player)	
		Head	*Tail*
Player 1 (row player)	*Head*	$(\underline{1}, -1)$	$(-1, \underline{1})$
	Tail	$(-1, \underline{1})$	$(\underline{1}, -1)$

Remark 47 *There are no Nash equilibrium. If player 1 plays Head, then the best choice for player 2 is to choose Tail. If player 2 plays Tail, then the best choice for player 1 is to choose Tail. If player 1 plays Tail, then the best choice for player 2 is to choose Head. If player 2 plays Head, then the best choice for player 1 is to choose Head. And so on. There is a cycle...*

13.8 Strict Domination

Definition 55 *Player i's action a_i* ***strictly dominates*** *her action a_i' if*

$$u_i(a_i, a_j) > u_i(a_i', a_j) \text{ for every action } a_j \text{ of player } j \neq i$$

We say that action a_i' is ***strictly dominated****.*

Remark 48 *A player's action strictly dominates another action if it is superior, no matter what the other player do. A strictly dominated action is not used in any Nash equilibrium.*

Remark 49 *In the Prisoner's Dilemma, the action Fink strictly dominates the action Quiet, for each player. To see this, suppose that player 2 player plays Quiet. Then, the best choice for player 1 is to choose Fink. If player 2 plays Fink, then the best choice for player 1 is to choose Fink. Thus, for player 1 Fink is superior to Quiet, no matter what the other player do. The same reasoning is true for player 2.*

Remark 50 *In BoS, neither action strictly dominates the other: Bach is better than Stravinsky if the other player chooses Bach, but is worse than Stravinsky if the other player chooses Stravinsky.*

13.9 Weak Domination

Definition 56 *Player i's action a_i* ***weakly dominates*** *her action a_i' if*

$$u_i(a_i, a_j) \geq u_i(a_i', a_j) \text{ for every action } a_j \text{ of player } j \neq i$$

and

$$u_i(a_i, a_j) > u_i(a_i', a_j) \text{ for some action } a_j \text{ of player } j$$

We say that action a_i' is ***weakly dominated****.*

Example 38 *Consider the following game where only the payoffs of player 1 are reported:*

		Player 2 (column player)	
		L	*R*
Player 1 (row player)	*T*	$1, -$	$0, -$
	M	$2, -$	$0, -$
	B	$2, -$	$1, -$

M weakly dominates T. B weakly dominates M. B strictly dominates T.