Technology

(Reading: Varian Chapter 19)

We start our discussion on **Firm Behaviour**.

First we remind ourselves some key concepts you would have learnt last year.

- A technology is a process by which inputs are converted to an output.

  E.g. labour, a computer, a projector, electricity, and software are being combined to produce this lecture.

- Usually several technologies will produce the same product -- a blackboard and chalk can be used instead of a computer and a projector.
- Which technology is "best"?
- How do we compare technologies?

Input Bundles

x_i is the amount used of input i; (level of input i)

An input bundle is a vector of the input levels;

$(x_1, x_2, \ldots, x_n)$.

Production Functions

The technology's production function states the maximum amount of output possible from an input bundle.

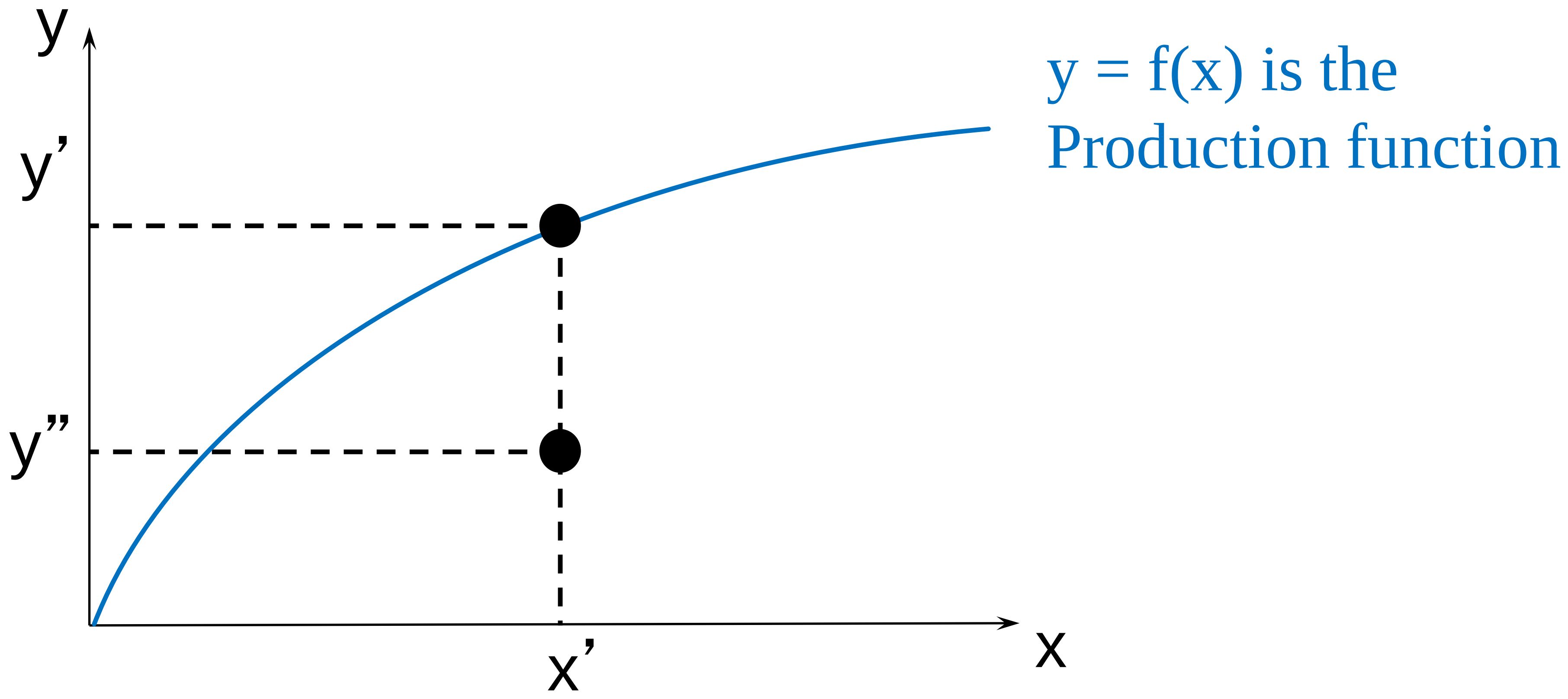
$$y \le f(x_1, \cdots, x_n)$$

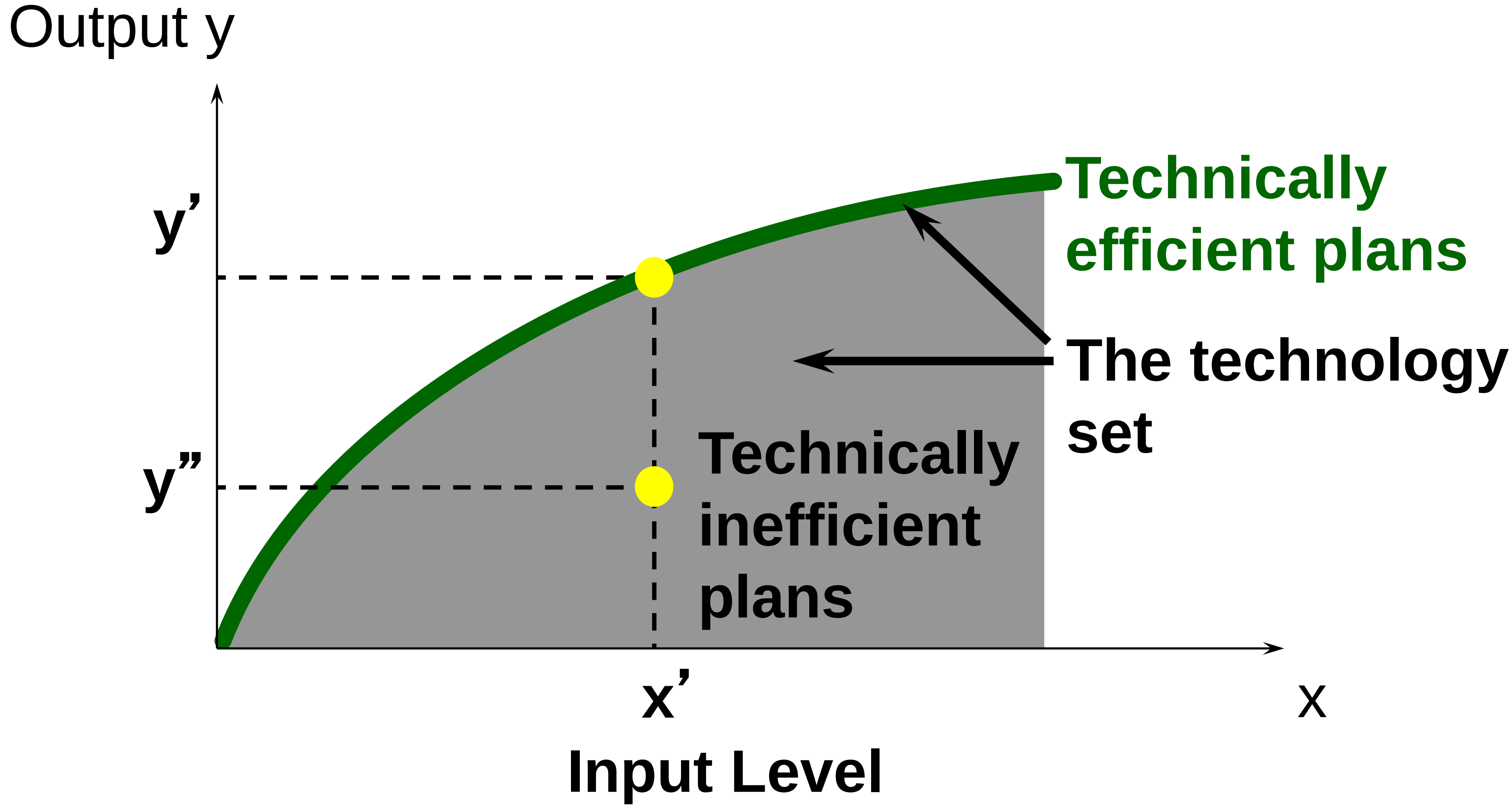
y denotes the output level.

Production Functions with One input (x), One output (y)

From x' input units, the maximal output obtainable is y'.

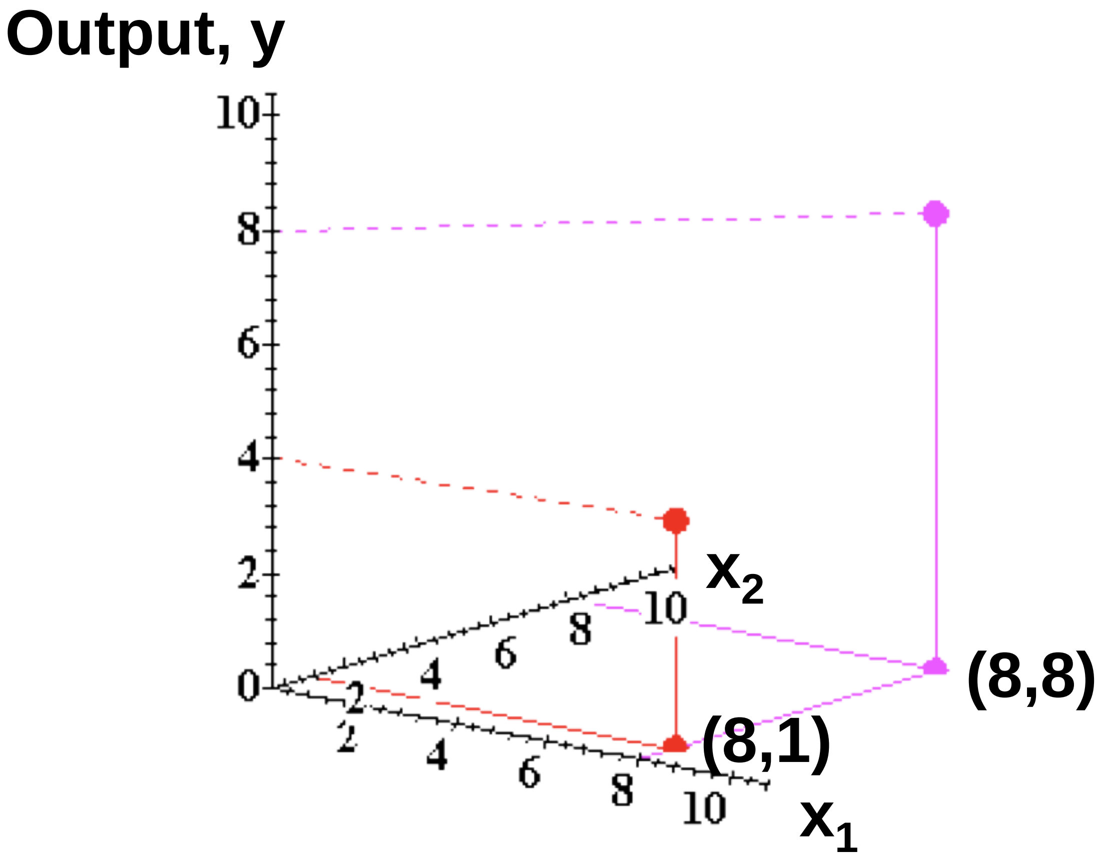
But, an output level that is feasible could be y'' = f(x'),.





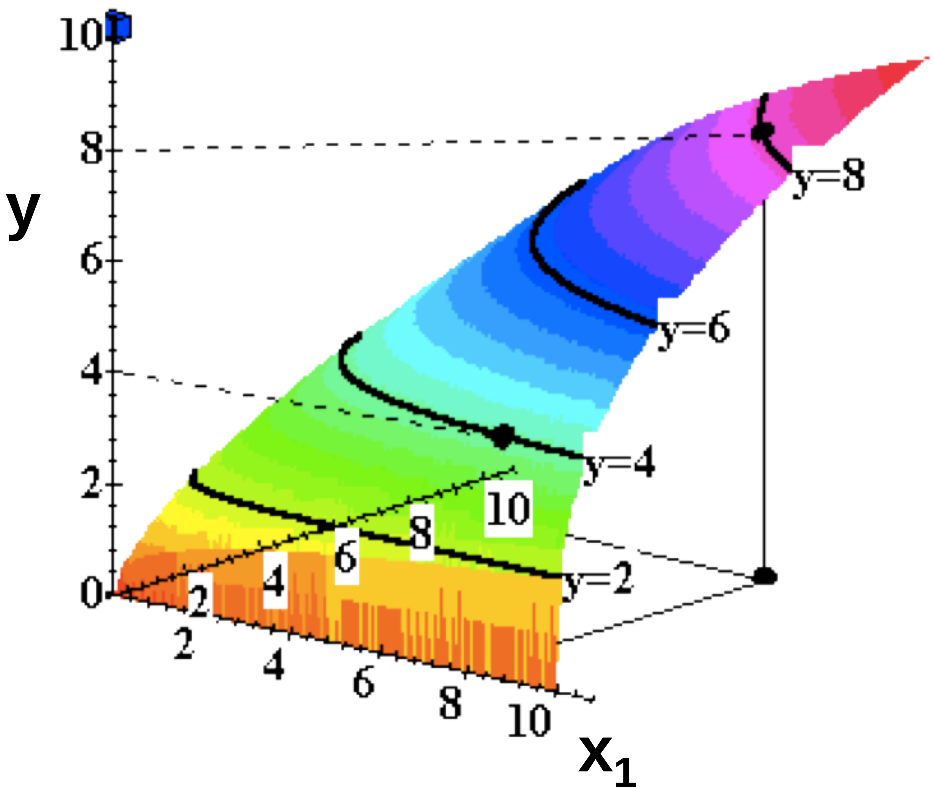
Output y
y’
y”
Technically efficient plans
The technology set
Technically inefficient plans
x’
x
Input Level

Technologies with Two Inputs



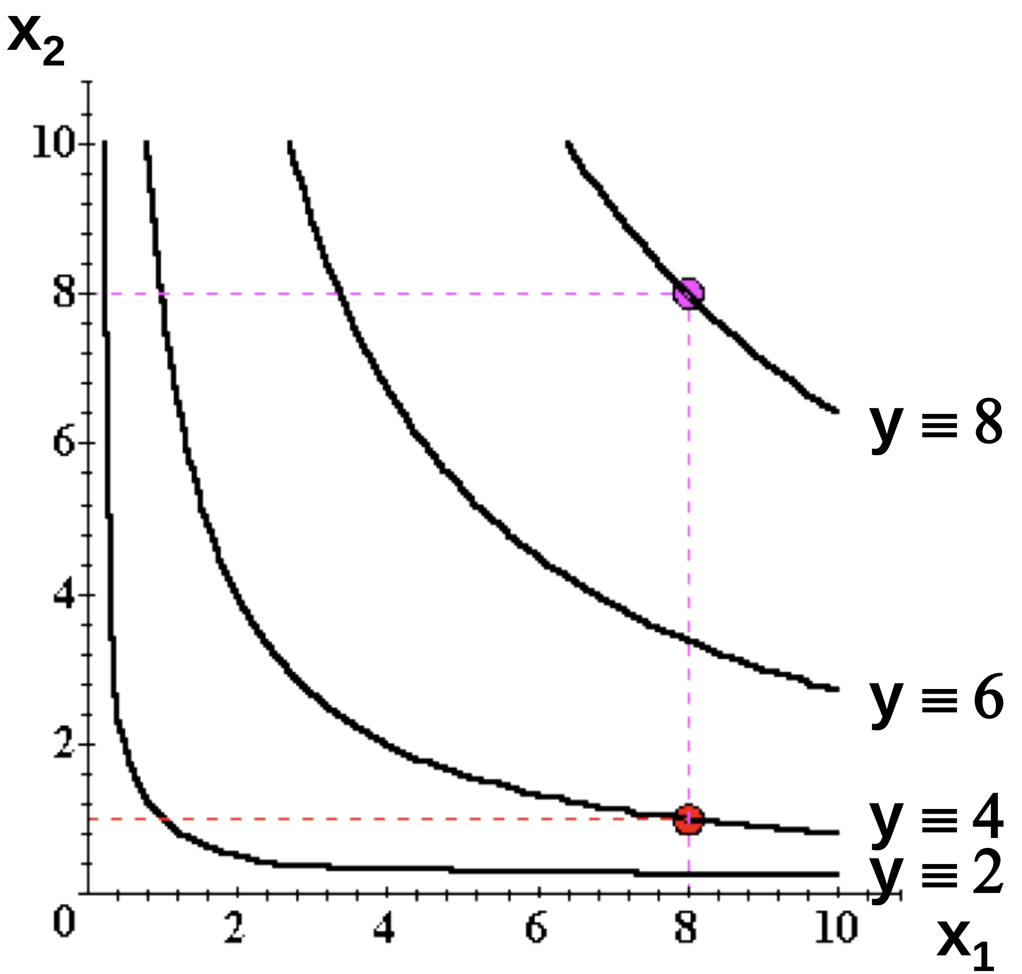
Isoquants

The y output unit isoquant is the set of all input bundles that yield at most the same output level y.



Isoquants can be graphed by adding an output level axis and displaying each isoquant at the height of the isoquant's output level.

Isoquants with Two Variable Inputs:



Some Production Technologies

- Perfect Substitution technology
- Perfect Complement (Fixed Proportion) technology
- Cobb-Douglas technology

Perfect-Substitutes Technologies

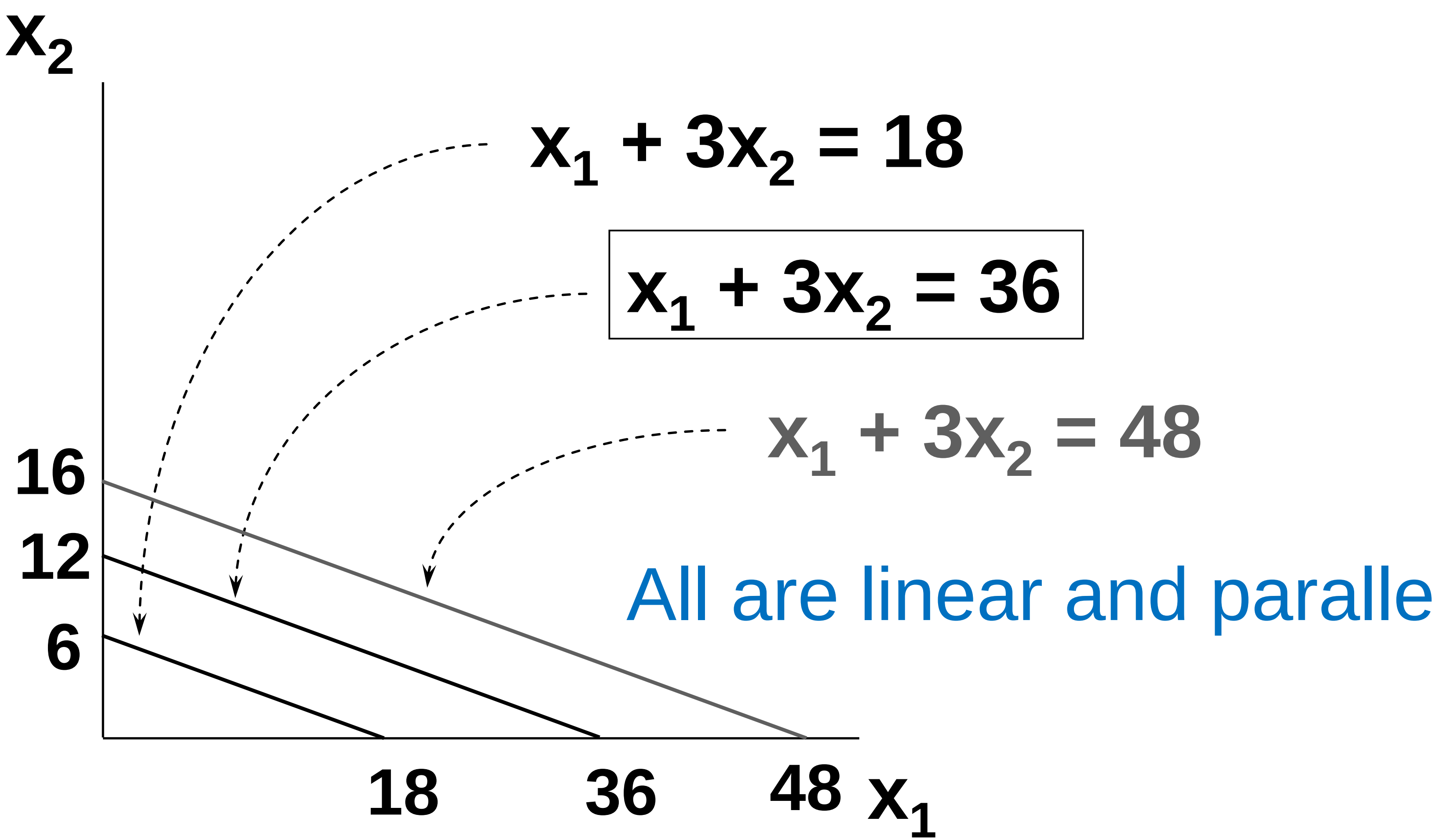
A perfect-substitutes production function is of the form

$$y = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n .$$

Notice that there can be production even with only one of the inputs. There is no need for all the inputs to be used.

E.g. $y = x_1 + 3x_2$

$$y = x_1 + 3x_2$$



x_2
$x_1 + 3x_2 = 18$
$x_1 + 3x_2 = 36$
$x_1 + 3x_2 = 48$
16
12
6
All are linear and parallel
18
36
48
x_1

Perfect Complement Technology

(Fixed-Proportions Technology)

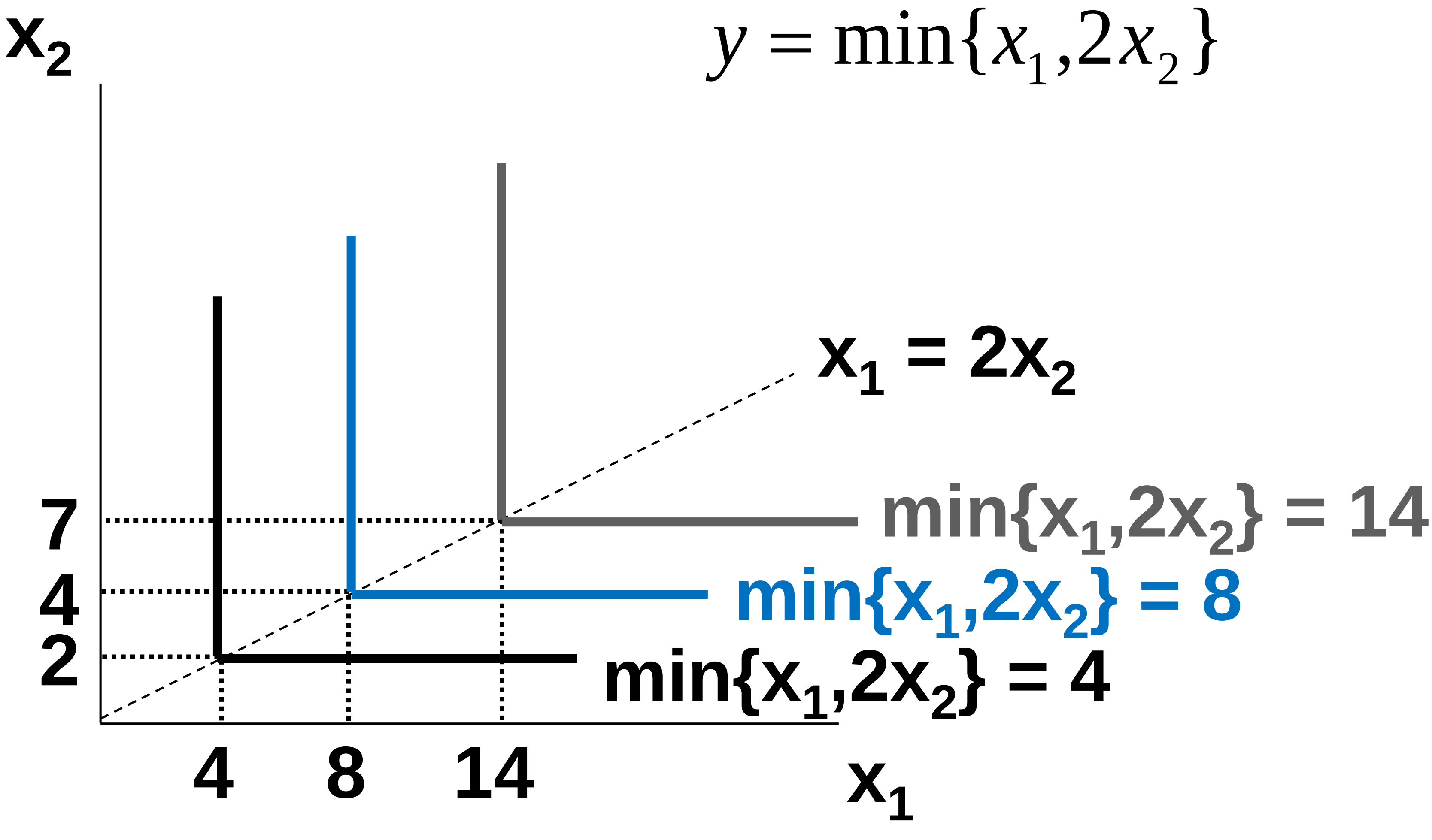
You need one input to complement the other, in order to be able to produce – i.e., both are needed at a minimum given proportion for production to occur)

A fixed-proportions production function is of the form

$$y = \min\{a_1 x_1, a_2 x_2, \cdots, a_n x_n\}.$$

E.g.

$$y = \min\{x_1, 2x_2\}$$



$y = \min\{x_1, 2x_2\}$
x_2
$x_1 = 2x_2$
7
4
2
$\min\{x_1,2x_2\} = 14$
$\min\{x_1,2x_2\} = 8$
$\min\{x_1,2x_2\} = 4$
4
8
14
x_1

Cobb-Douglas Technologies

A Cobb-Douglas production function is of the form

$$y = Ax_1^{a_1} x_2^{a_2} \cdots\cdots x_n^{a_n}.$$

with $0 < a_m < 1,\ m = 1..n$

E.g. $y = x_1^{1/3} x_2^{1/3}$

Give more eg.s of Cobb-Douglas production functions.

What more can you say about Cobb-Douglas technology?

Marginal (Physical) Products

Consider a production function $y = f(x_1, \cdots, x_n)$

The marginal product of input i is

the rate-of-change of the output level as the level of input i changes, holding all other input levels fixed.

That is,

$$MP_i = \frac{\partial y}{\partial x_i}$$

Eg: Consider a Cobb-Douglas production function

$$y = f(x_1, x_2) = x_1^{1/3} x_2^{2/3}$$

- marginal product of input 1 is

$$MP_1 = \frac{\partial y}{\partial x_1} = \frac{1}{3} x_1^{-2/3} x_2^{2/3}$$

- marginal product of input 2 is

$$MP_2 = \frac{\partial y}{\partial x_2} = \frac{2}{3} x_1^{1/3} x_2^{-1/3}.$$

Notice that both MP_1 and MP_2 depend on x_1 as well as x_2

Diminishing Marginal Product

The marginal product of input *i* is diminishing if it becomes smaller as the level of input i increases.

$$\frac{\partial MP_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial y}{\partial x_i}\right) = \frac{\partial^2 y}{\partial x_i^2} < 0.$$

$$y = x_1^{1/3} x_2^{2/3}$$

$$MP_1 = \frac{\partial y}{\partial x_1} = \frac{1}{3} x_1^{-2/3} x_2^{2/3}$$

$$MP_2 = \frac{\partial y}{\partial x_2} = \frac{2}{3} x_1^{1/3} x_2^{-1/3}$$

$$\frac{\partial MP_1}{\partial x_1} = \frac{\partial^2 y}{\partial x_1^2} = -\frac{2}{9} x_1^{-5/3} x_2^{2/3} < 0$$

$$\frac{\partial MP_2}{\partial x_2} = \frac{\partial^2 y}{\partial x_2^2} = -\frac{2}{9} x_1^{1/3} x_2^{-4/3} < 0.$$

Both marginal products are diminishing.

Returns-to-Scale

We discussed that,

The Marginal Products describe the change in output level as a single input level changes, holding the other inputs constant.

Returns-to-scale describes how the output level changes when all input levels change in direct proportion (*e.g.* all input levels doubled, or halved).

Constant returns-to-scale

If, for any input bundle $(x_1,\ldots,x_n)$,

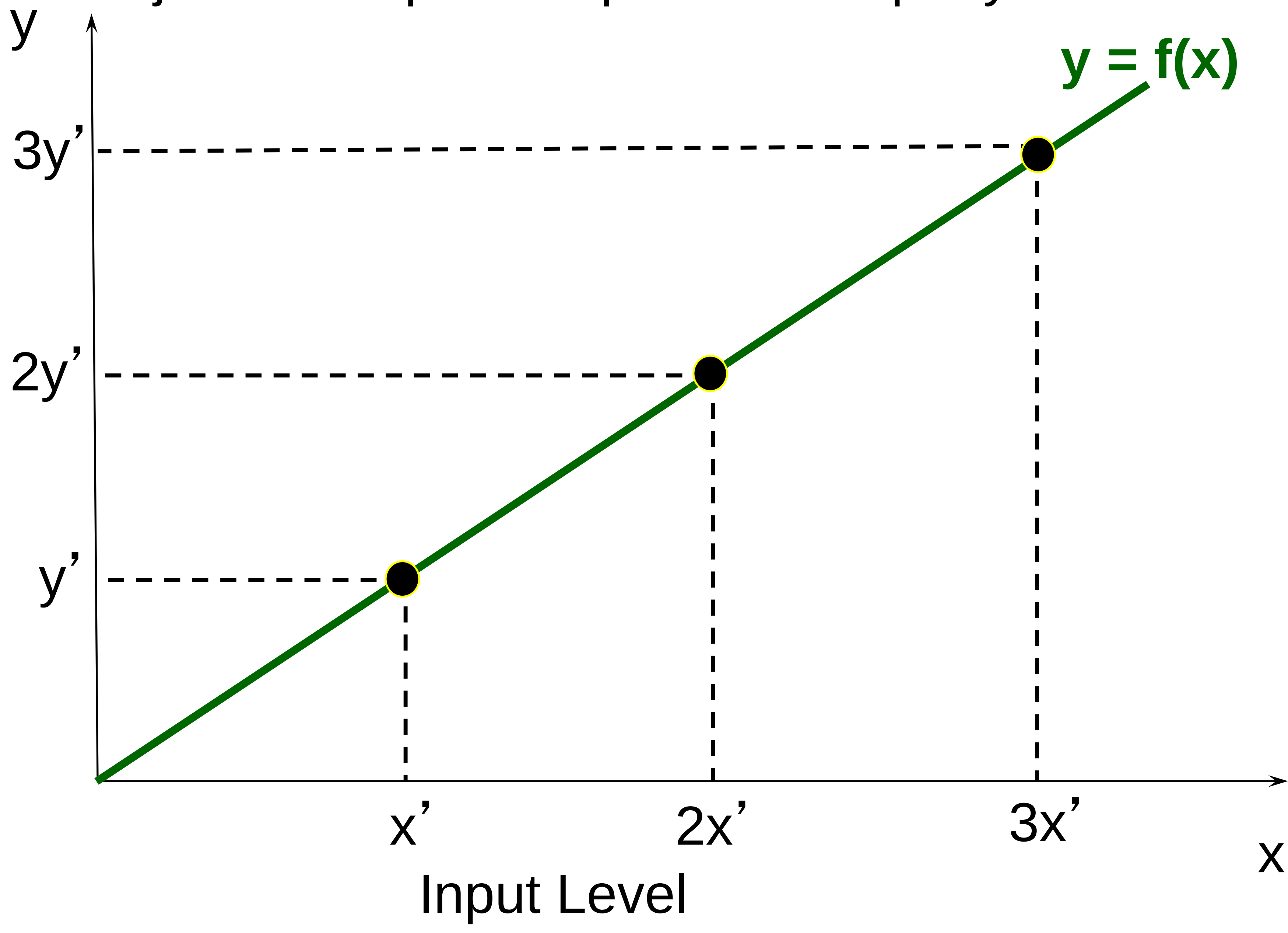
$$f(kx_1, kx_2, \cdots, kx_n) = kf(x_1, x_2, \cdots, x_n)$$

production function f exhibits constant returns-to-scale.

E.g.

doubling all input levels (k = 2) ➔ doubles output level.

Lets have just one input x to produce output y



Diminishing returns-to-scale

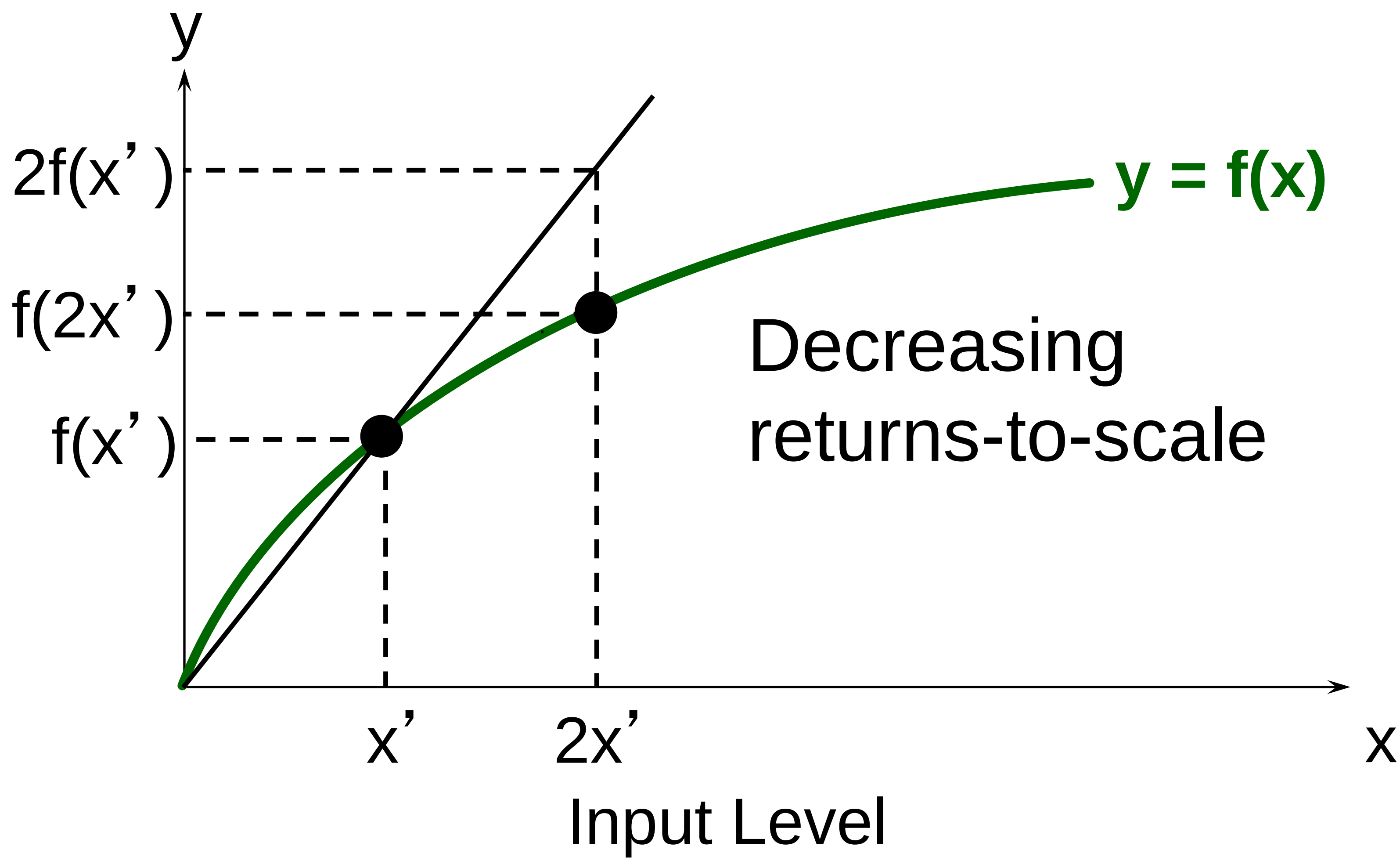
If, for any input bundle $(x_1,\ldots,x_n)$,

$$f(kx_1, kx_2, \cdots, kx_n) < kf(x_1, x_2, \cdots, x_n)$$

then the technology exhibits diminishing returns-to-scale.

E.g.

doubling all input levels (k = 2) less than doubles output.


y
2f(x')
f(2x')
f(x')
y = f(x)
Decreasing
returns-to-scale
x'
2x'
x
Input Level

Increasing returns-to-scale.

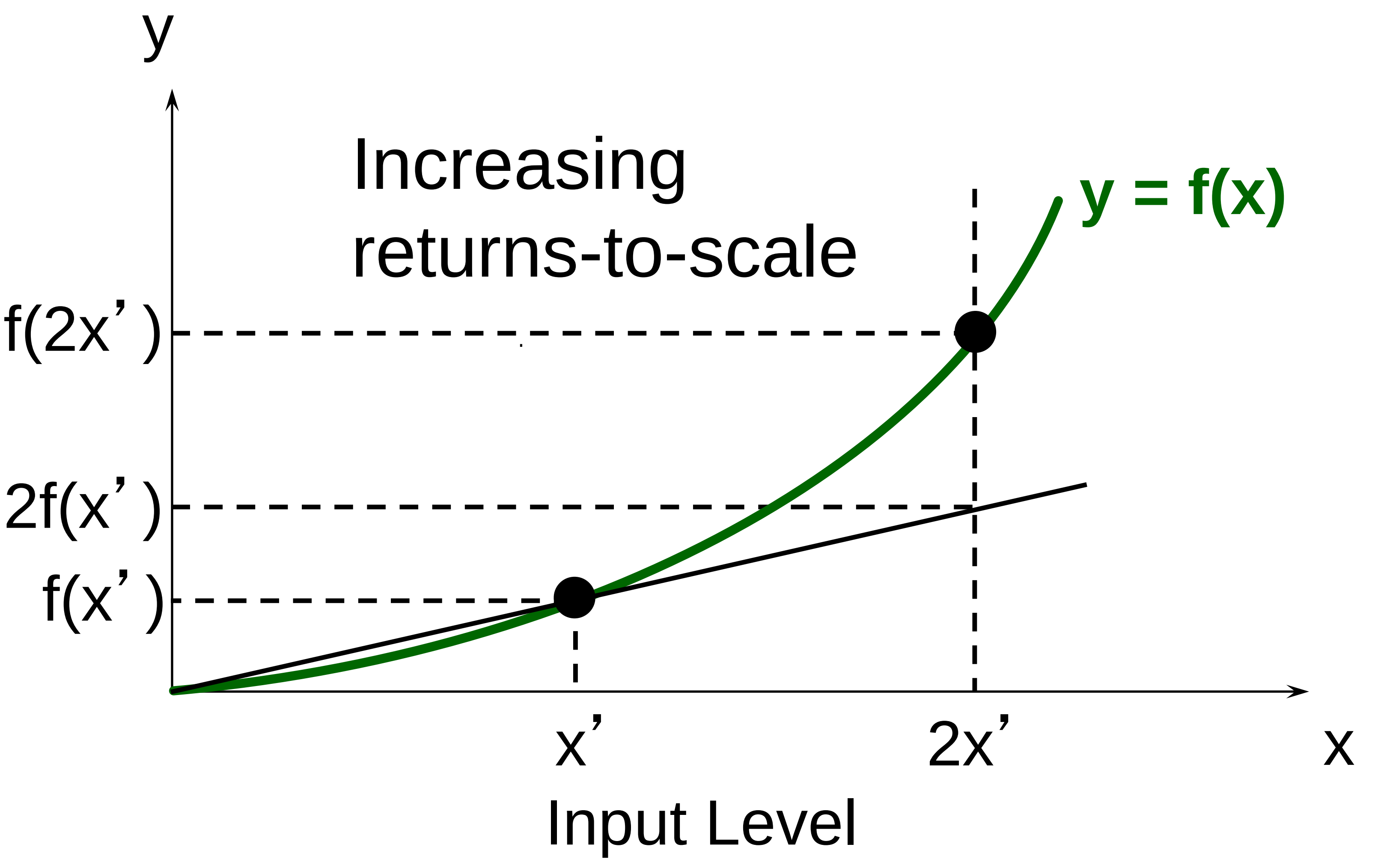
If, for any input bundle $(x_1,\ldots,x_n)$,

$$f(kx_1, kx_2, \cdots, kx_n) > kf(x_1, x_2, \cdots, x_n)$$

then the technology exhibits increasing returns-to-scale.

E.g. doubling all input levels (k = 2)

➔ more than doubles the output level.


y
Increasing
returns-to-scale
y = f(x)
f(2x’)
2f(x’)
f(x’)
x’
2x’
x
Input Level

Examples of Returns-to-Scale

The perfect-substitutes production function is

$$y = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

If we expand all input levels proportionate y by k.

➔ the output level becomes

$$\begin{aligned} & a_1(kx_1) + a_2(kx_2) + \cdots + a_n(kx_n) \\ &= k(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n) \\ &= ky. \end{aligned}$$

The perfect-substitutes production function exhibits constant returns-to-scale.

Eg: $y = 6x_1 + 500x_2 + 500x_2 + 300x_4$

If we double all input levels,

$y' = 12x_1 + 1000x_2 + 1000x_2 + 600x_4$
= 2 $(6x_1 + 500x_2 + 500x_2 + 300x_4)$
= 2y

Eg: $y = ax_1 + bx_2$

If we increase all input levels by k.

$y' = akx_1 + bkx_2$
= k $(ax_1 + bx_2)$
= ky

constant returns-to-scale.

The Cobb-Douglas production function

$$y = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

If we expand all input levels proportionately by k.

➔ the output level becomes

$$(kx_1)^{a_1}(kx_2)^{a_2}\cdots(kx_n)^{a_n} = k^{a_1}k^{a_2}\cdots k^{a_n} x^{a_1} x^{a_2} \cdots x^{a_n}$$

$$= k^{a_1+a_2+\cdots+a_n} \underbrace{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}$$

$$= k^{a_1+\cdots+a_n} y.$$

The Cobb-Douglas technology is

constant returns-to-scale if $a_1 + \ldots + a_n = 1$

increasing returns-to-scale if $a_1 + \ldots + a_n > 1$

decreasing returns-to-scale if $a_1 + \ldots + a_n < 1$.

With only two inputs, $y = x_1^a x_2^b$

If we expand all input levels proportionately by k.

➔ the output level becomes

$$(kx_1)^a (kx_2)^b = k^a k^b x_1{}^a x_2{}^b$$
$$= k^{a+b} x_1^a x_2^b$$
$$= k^{a+b} y.$$

Notice:

constant returns-to-scale if a+b = 1

increasing returns-to-scale if a + b > 1

decreasing returns-to-scale if a + b < 1.

Technical Rate-of-Substitution

The rate that a firm can substitute one input for another

without changing its output level.

The slope of an isoquant is its technical rate-of-substitution.

The slope is the rate at

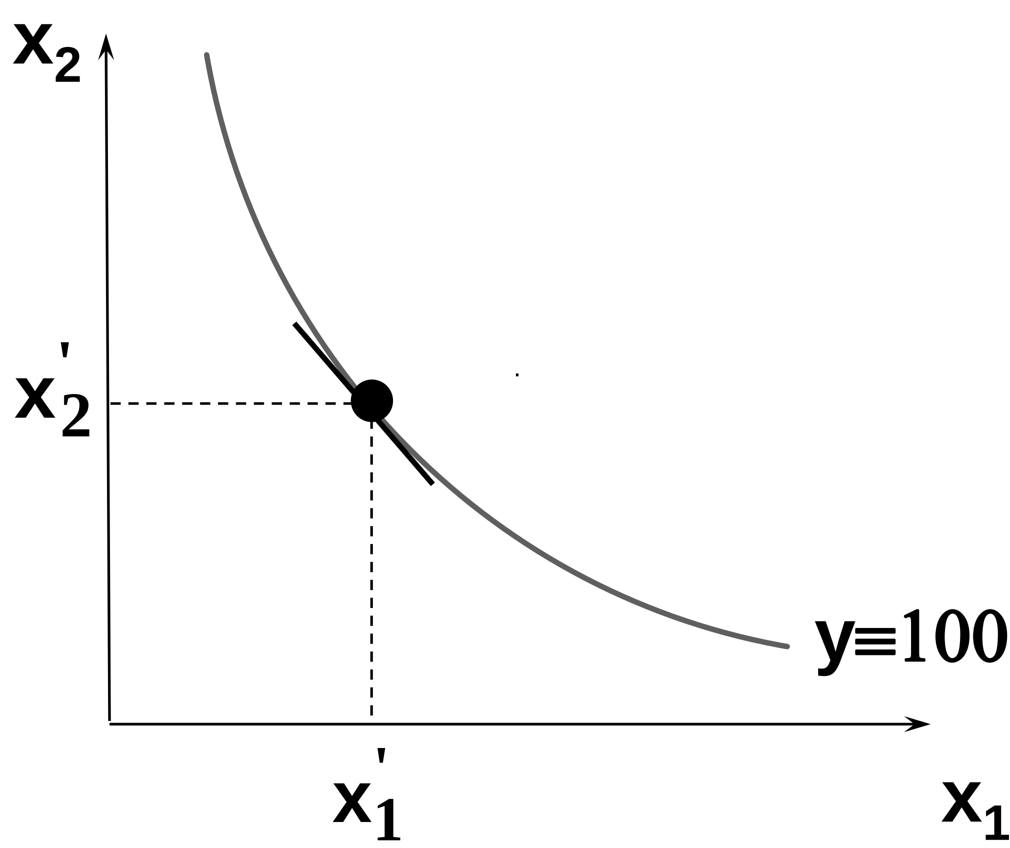
which input 2 must be

given up

as input 1's level is

increased so as not to

change the output level.



Computing technical rate-of-substitution

Let the production function be $y = f(x_1, x_2).$

A small change (dx_1, dx_2) in the input bundle causes a change to the output level of

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 .$$

If there is to be no change to the output level, dy = 0 ,

so the changes dx_1 and dx_2 to the input levels must satisfy

$$0 = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 .$$

$$0 = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2$$

rearranges to

$$\frac{\partial y}{\partial x_2} dx_2 = -\frac{\partial y}{\partial x_1} dx_1$$

$$\boxed{\frac{dx_2}{dx_1} = -\frac{\partial y / \partial x_1}{\partial y / \partial x_2}.}$$

This is the slope of the isoquant

– i.e. which is the rate at which input 2 must be given up as

input 1 increases so as to keep the output level constant.

A Cobb-Douglas Example

$$y = f(x_1, x_2) = x_1^a x_2^b$$

$$\frac{\partial y}{\partial x_1} = a x_1^{a-1} x_2^b \qquad \frac{\partial y}{\partial x_2} = b x_1^a x_2^{b-1}.$$

The technical rate-of-substitution is

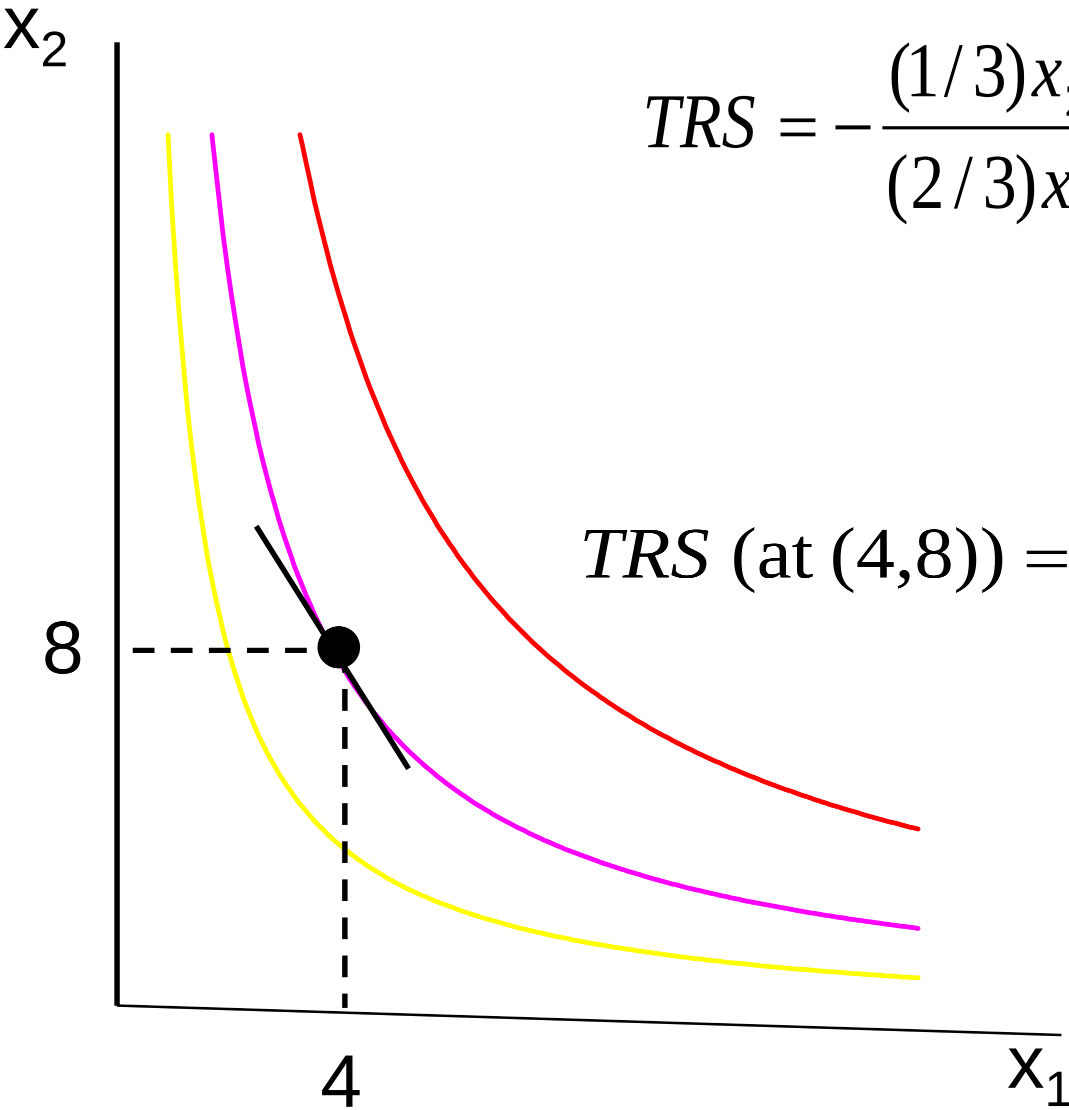
$$\frac{dx_2}{dx_1} = -\frac{\partial y / \partial x_1}{\partial y / \partial x_2}$$

$$= -\frac{a x_1^{a-1} x_2^b}{b x_1^a x_2^{b-1}} = -\frac{a x_2^{b-b+1}}{b x_1^{a-a+1}}$$

$$= -\frac{a x_2}{b x_1}.$$

$$\boxed{\frac{dx_2}{dx_1} = -\frac{a x_2}{b x_1}.}$$

$$y = x_1^{1/3} x_2^{2/3}$$



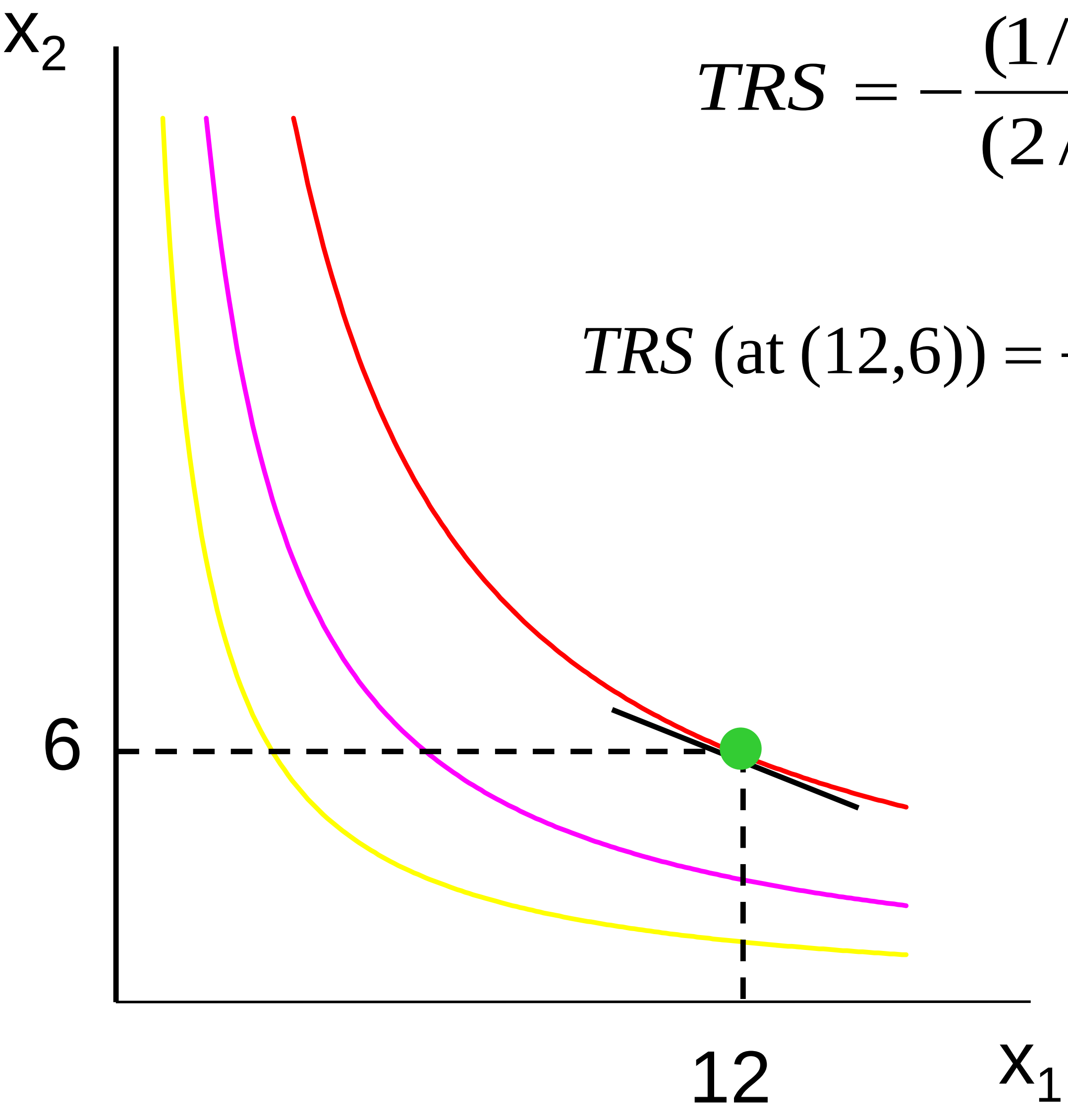
$$TRS = -\frac{(1/3)x_2}{(2/3)x_1} = -\frac{x_2}{2x_1}$$

$$TRS \text{ (at } (4,8)) = -\frac{x_2}{2x_1} = -\frac{8}{2 \times 4} = -1$$

$$y = x_1^{1/3} x_2^{2/3}$$

$$TRS = -\frac{(1/3)x_2}{(2/3)x_1} = -\frac{x_2}{2x_1}$$

$$TRS \text{ (at (12,6))} = -\frac{x_2}{2x_1} = -\frac{6}{2 \times 12} = -\frac{1}{4}$$



Well-Behaved Production Technologies

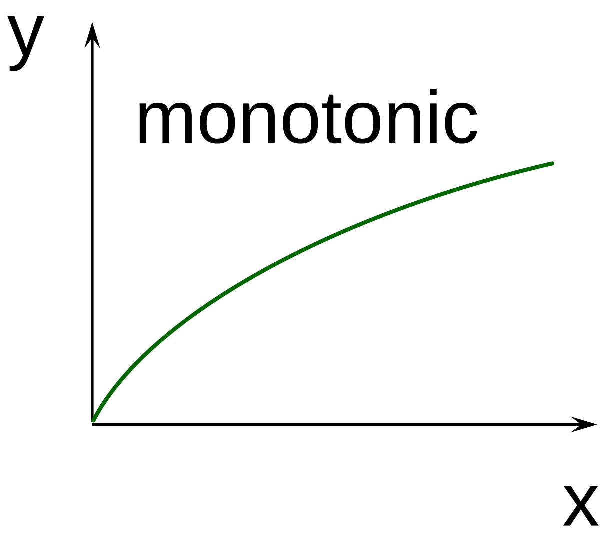
A well-behaved technology is

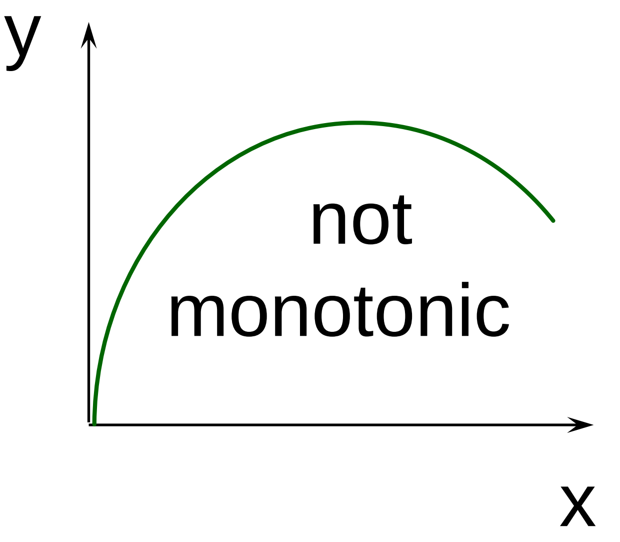
- Monotonic
- Convex.

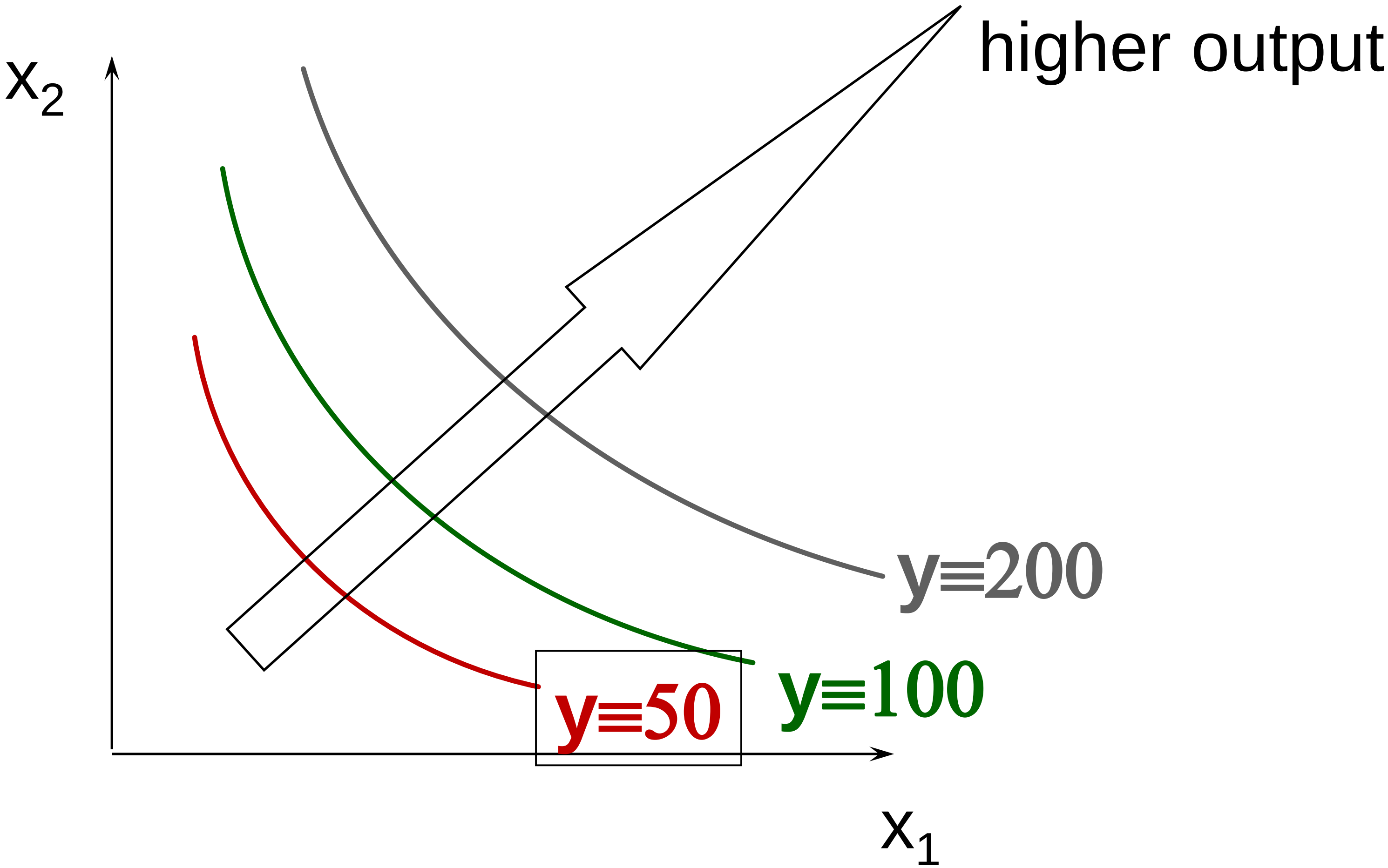
What does this mean and is it reasonable?

Monotonicity

Monotonicity: More of any input generates more output.







higher output
x2
x1
y≡200
y≡100
y≡50

Convexity implies ➔ TRS increases (becomes less negative) as x_1 (input in the x-axis) increases.