Chapter 4: Introduction to Probability

Probability: numerical measure of the likelihood that an event will occur.

Probabilities values are always assigned on a scale from 0 to 1. A probability near zero indicates an event is unlikely to occur; a probability near 1 indicates an event is almost certain to occur. A .50 probability indicates the occurrence of the event is just as likely as it is unlikely. Other probabilities between 0 and 1 represent other degrees of likelihood that an even will occur.

4.1 Random Experiments, Counting Rules, and Assigning Probabilities

Random experiment: process that generates well-defined experimental outcomes. On any single repetition or trial of the experiment, the one possible outcome that occurs is determined by chance.

Sample space (S): set of all experimental outcomes (or sample points).

e.g.: Random experiment of tossing a coin, with possible outcomes defined as the upward face of the coin, S = {Head, Tail}.
e.g.: Random experiment of rolling a die, with possible outcomes defined as the # of dots on the upward face of the die, S = {1, 2, 3, 4, 5, 6}.

Counting rule for multiple-step experiments: If an experiment can be described as a sequence of k steps with n_1 possible outcomes on the first step, n_2 possible outcomes on the second step, and so on, then the total number of experimental outcomes is given by $(n_1)(n_2) \ldots (n_k)$.

Tree diagram: graphical representation that helps in visualizing a multiple-step experiment.

Counting rule for combinations: number of combinations of n objects from a set of N, n ≤ N.

$$C_n^N = \binom{N}{n} = \frac{N!}{n!\,(N-n)!}$$

where "N factorial", N! = N(N − 1)(N −2) … (2)(1). n! is defined similarly and 0! = 1

Counting rule for permutations: number of permutations of n objects from a set of N, n ≤ N—when the order of selection of the objects matters.

$$P_n^N = n!\binom{N}{n} = \frac{N!}{(N-n)!}$$

An experiment results in more permutations than combinations for the same number of objects because every selection of *n* objects can be ordered in *n*! different ways.

Assigning Probabilities: Let E_i be the *i*th experimental outcome and $P(E_i)$ its probability, then it must be that, $0 \leq P(E_i) \leq 1$ for all *i*, and $\Sigma P(E_i) = 1$.

Assign a probability of 1/n to each experimental outcome, when all are equally likely (**classical method**). If data are available, estimate the proportion of the time each experimental outcome occurs when the experiment is repeated a large number of times (**relative frequency method**). When outcomes are not equally likely and little relevant data are available, use experience or intuition to assign probabilities that expresses your degree of belief that experimental outcomes will occur (**subjective method**).

4.2 Events and Their Probabilities

Event: a collection of sample points.

Probability of an event: is equal to the sum of the probabilities of the sample points in the event.

4.3 Some Basic Relationships of Probability

Complement of event A: is the event, denoted A^c, consisting of all sample points that are *not* in A.
$P(A) + P(A^c) = 1$

Intersection of events A & B (A∩B): is the event containing the sample points belonging to *both* A *and* B.
Union of events A & B (AUB): is the event containing *all* sample points belonging to A *or* B *or* both.
Addition law: P(AUB) = P(A) + P(B) − P(A∩B), where P(AUB) is the probability that A *or* B *or* both occur, i.e. probability that at least one of the two events occur.

NOTE: **Joint probabilities**, such as P(A∩B), give the probabilities of the intersection of two events.

Mutually exclusive events: have no sample points in common. In other words, when one event occurs, the other cannot occur. In this case, P(A∩B) = 0 and P(AUB) = P(A) + P(B).

4.4 Conditional Probability

Conditional probability: probability of event A *given* the condition that event B has occurred.
P(A|B) = P(A∩B)/P(B) when A and B are dependent events
P(A|B) = P(A) when A and B are **independent** events

NOTE: Joint probability Table 4.5 is similar to Table 2.10b Crosstabulation of Quality Rating & Mean Price_relative frequency (see Ch.2.3 Presentation)

Multiplication law:
P(A∩B) = P(B)P(A|B) or P(A∩B) = P(A)P(B|A) when A and B are dependent events
P(A∩B) = P(A)P(B) when A and B are **independent** events

Two events with nonzero probabilities cannot be both mutually exclusive and independent. If one of two mutually exclusive events is known to occur, the other cannot occur; thus, the probability of the other event occurring is zero. They are therefore dependent.

4.5 Bayes' Theorem

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n)}$$

where, **posterior probability** $P(A_i|B)$ is calculated using **prior probabilities** $P(A_1)$, $P(A_2)$, …, $P(A_n)$ when the events are mutually exclusive and their union is the entire sample space. See Equations (4.14)-(4.16) for proof, for the two-event case. Note, if the union of events is the entire sample space, the events are said to be collectively exhaustive.

Consider a manufacturing firm that receives shipments of parts from two different suppliers.
A_1: event that a part is from supplier 1; $P(A_1) = .65$.
A_2: event that a part is from supplier 2; $P(A_2) = .35$.
65% of the parts purchased by the company are from supplier 1 and the remaining 35% are from supplier 2.

The quality of the purchased parts varies with the source of supply.
G: event that a part is good; $P(G|A_1) = .98$ and $P(G|A_2) = .95$.
B: event that a part is bad; $P(B|A_1) = .02$ and $P(B|A_2) = .05$.
98% of the parts purchased from supplier 1 are good and 2% are bad.
(In other words, given information the part is from supplier 1, the probability that the part is good is .98 and the probability that the part is bad is 0.02.)
95% of the parts purchased from supplier 2 are good and 5% are bad.
(In other words, given information the part is from supplier 2, the probability that the part is good is .95 and the probability that the part is bad is 0.05.)

Using the multiplication law for dependent events, we construct the following joint probability table or crosstabulation of supplier & part quality (similar to Table 4.5 or Table 2.10b from Ch.2.3 Presentation). Table A shows the details of the computations. Table B shows the results.

Table A

		Good	Bad	TOTAL
Supplier	1	$P(A_1 \cap G) = P(A_1)P(G|A_1)$ $= .65(.98)$	$P(A_1 \cap B) = P(A_1)P(B|A_1)$ $= .65(.02)$	
	2	$P(A_2 \cap G) = P(A_2)P(G|A_2)$ $= .35(.95)$	$P(A_2 \cap B) = P(A_2)P(B|A_2)$ $= .35(.05)$	
	TOTAL			

Table B

		Good	Bad	TOTAL
Supplier	1	**0.6370**	**0.0130**	0.65
	2	**0.3325**	**0.0175**	0.35
	TOTAL	0.9695	0.0305	1.00

Next, we use Table B to compute the following conditional or posterior probabilities:
$P(A_1|G) = P(A_1 \cap G)/P(G) = 0.6370/0.9695 = 0.6570$
$P(A_2|G) = P(A_2 \cap G)/P(G) = 0.3325/0.9695 = 0.3430$; Note: $P(A_1|G) + P(A_2|G) = 1.00$
Given information that the part is good, the probability that the part is from supplier 1 is .6570 and the probability that the part is from supplier 2 is .3430.
$P(A_1|B) = P(A_1 \cap B)/P(B) = 0.0130/0.0305 = 0.4262$
$P(A_2|B) = P(A_2 \cap B)/P(B) = 0.0175/0.0305 = 0.5738$; Note: $P(A_1|B) + P(A_2|B) = 1.00$
Given information that the part is bad, the probability that the part is from supplier 1 is .42620 and the probability that the part is from supplier 2 is .5738.