The University of Nottingham

School of Economics

Economics Tutorials in Nottingham

Calculus Revision Lecture

Lecture Overview

- Last lecture: representation of consumers' preferences using indifference curves.
- Preferences can also be represented mathematically using *utility functions*.
- Allows additional insights into consumer behaviour but requires use of calculus.
- This lecture:
  - Brief revision of basic concepts/rules for derivatives.
  - Introduction to partial derivatives.
- Next lecture: utility functions.

Overview

1. Functions of one and several variables.
2. Definition of derivatives.
3. Rules for calculating derivatives.
4. Finding maxima and minima of functions of one variable using calculus.
5. Functions of several variables: derivatives, finding maxima/minima, total differentials.

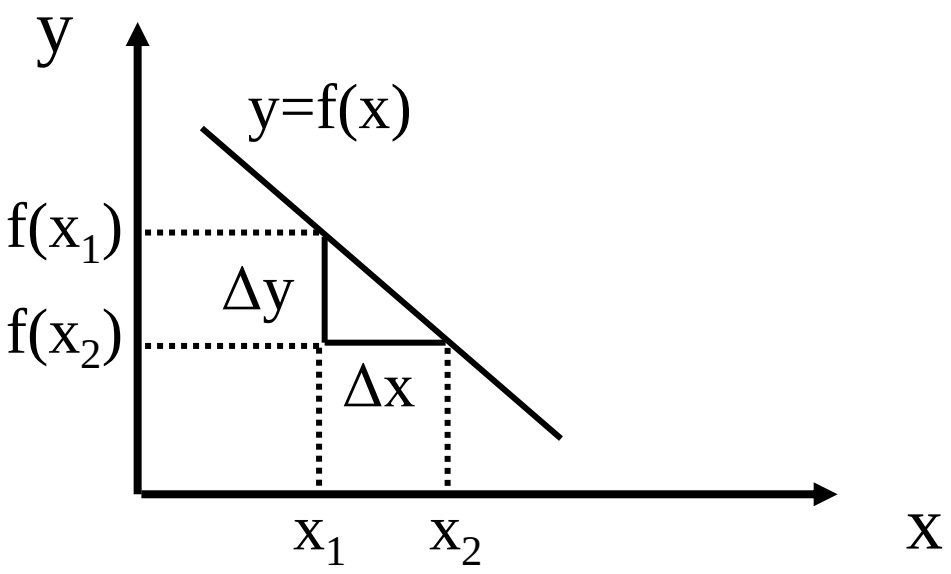
Functions

- A *function* associates each member of a set with a single member of another set.
  - Demand functions associate each price with a single corresponding quantity (e.g., $Q = 10 - p$)
  - Notation: $y = f(x)$ or similar.
- A function can depend on more than one variable.
  - Demand functions can also depend on the price of competing products (e.g., $Q_A = 10 - p_A + p_B$).
  - Notation: $z = f(x, y)$ or similar.
- Will first look at derivatives of functions of one variable and then of several variables.

Definition of Derivatives

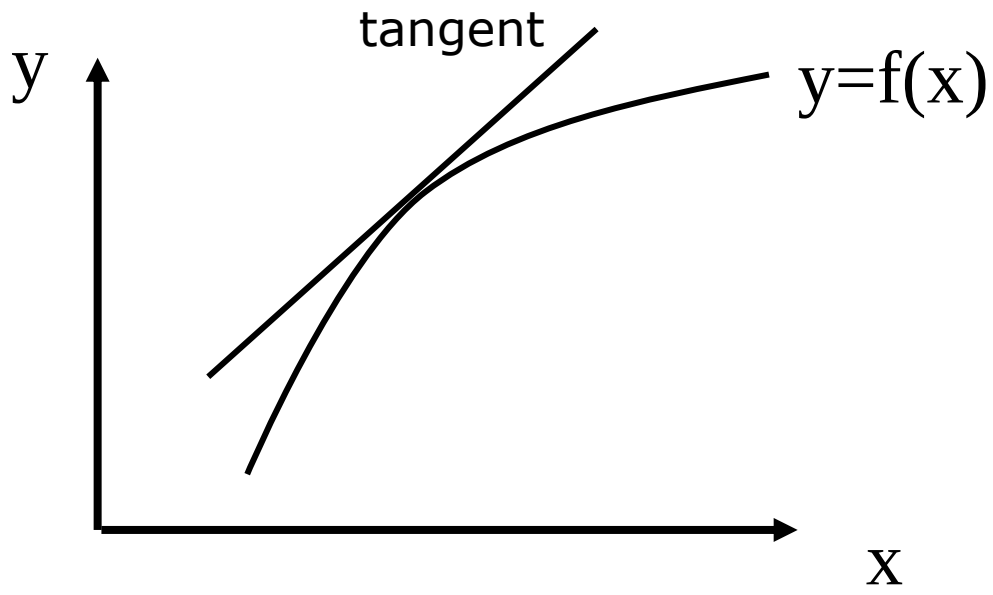
The slope of a linear function $y = f(x)$ is defined as:

$$slope = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



Definition of Derivatives

- The slope of a linear function is constant.
- The slope of a non-linear function at a given point is the slope of the tangent line at that point.
  - Tangent line: straight line that touches the curve at only one point.



Definition of Derivatives

- The derivative of a function $f(x)$ at a given point measures the slope of the tangent at that point.
- How to find that derivative/slope?
- Slope of a line between two points on the graph of $f(x)$:

$$slope_{x_1+h,x_1} = \frac{f(x_1 + h) - f(x_1)}{h}$$

- The slope of this line converges to the derivative at point x_1 as h approaches zero.

Definition of Derivatives

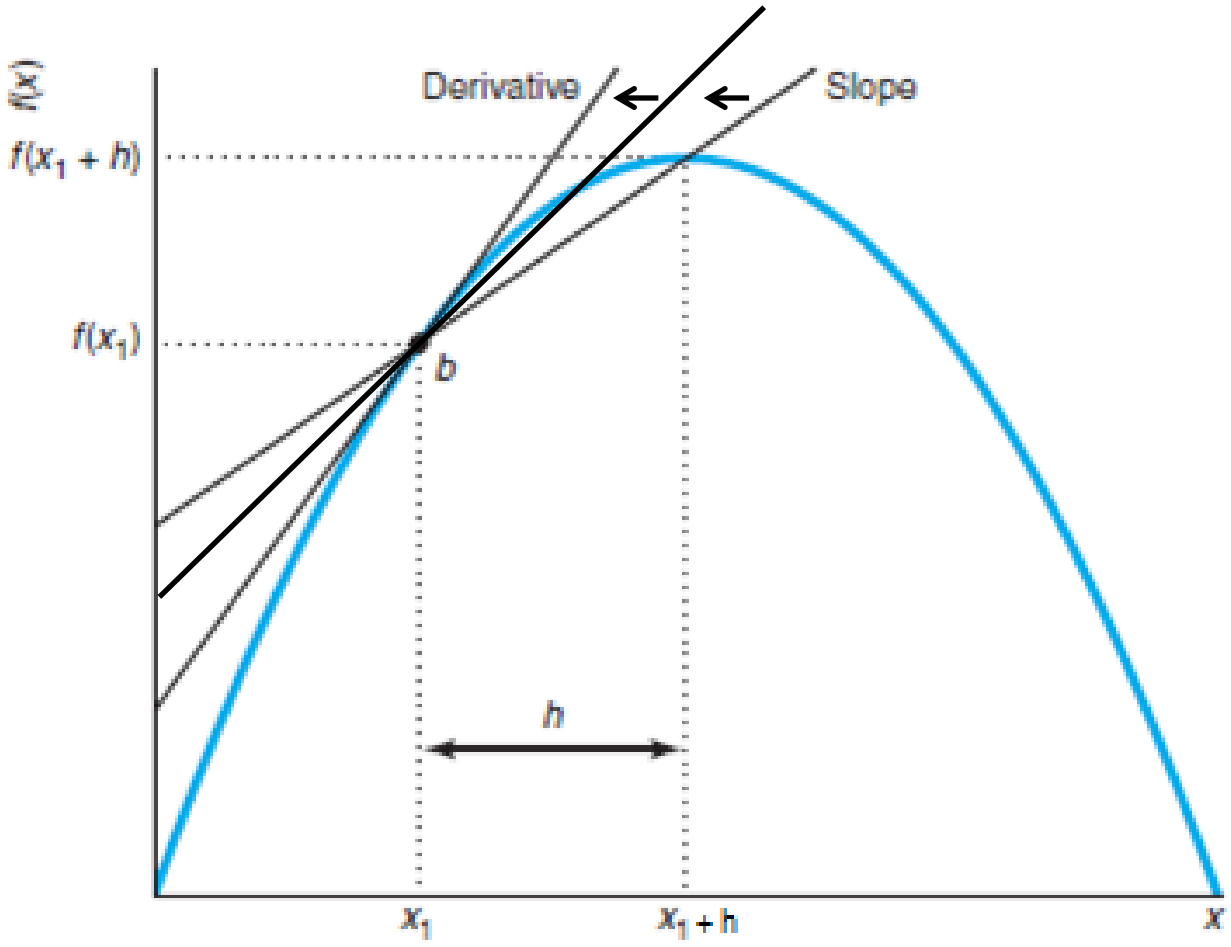


Figure A.2 from Perloff (2014)

Definition of Derivatives

- Formally, the derivative of the function $f(x)$ at point x is defined as:

$$\frac{\mathrm{d}f(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Other forms of notation for the derivative of $\mathrm{y} = f(x)$:

$$f'(x) \qquad \frac{dy}{dx} \qquad \frac{\mathrm{d}}{dx} f(x)$$

Rules for Calculating Derivatives

- Can use the definition formula from the last slide to derive rules for calculating derivatives.
- Constant-function rule: if $f(x) = a$, then $\mathrm{d}f(x)/dx = 0$.
- Power rule: if $f(x) = ax^b$, then $\mathrm{d}f(x)/dx = abx^{b-1}$.
  - Example: $f(x) = 2x^2$, $\mathrm{d}f(x)/dx = 4x$.
- Addition rule: if $f(x) = g(x) + h(x)$, then $\frac{\mathrm{d}f(x)}{dx} = \frac{\mathrm{d}g(x)}{dx} + \frac{\mathrm{d}h(x)}{dx}$.
  - Example: $\mathrm{f}(x) = x^2 + 2x$, then $\frac{\mathrm{d}f(x)}{dx} = 2x + 2$.

Rules for Calculating Derivatives

- Product rule: if $f(x) = g(x)h(x)$, then $\frac{\mathrm{d}f(x)}{dx} = \frac{\mathrm{d}g(x)}{dx}h(x) + g(x)\frac{\mathrm{d}h(x)}{dx}$.
  - Example: $f(x) = (2x+1)x^2$, $\frac{\mathrm{d}f(x)}{dx} = 2x^2 + (2x+1)2x$.
- Quotient rule: if $f(x) = g(x)/h(x)$, then $\frac{\mathrm{d}f(x)}{dx} = \frac{\frac{\mathrm{d}g(x)}{dx}h(x) - g(x)\frac{\mathrm{d}h(x)}{dx}}{[h(x)]^2}$.
  - Example: $f(x) = \frac{(2x+1)}{x^2}$, $\frac{\mathrm{d}f(x)}{dx} = \frac{2x^2-(2x+1)2x}{x^4}$

Rules for Calculating Derivatives

- Chain rule: if $f(x) = g\big(h(x)\big)$, then $\frac{\mathrm{d}f(x)}{dx} = \frac{\mathrm{d}g\big(h(x)\big)}{dh(x)}\frac{\mathrm{d}h(x)}{dx}$.
  - Example: $f(x) = (1 + x^2)^3$.
  - Could multiply out $(1 + x^2)(1 + x^2)\ (1 + x^2)$ and use the power rule.
  - But easier to apply chain rule:
    - $\mathrm{h}(x) = 1 + x^2$ and $f(x) = g\big(h(x)\big) = \big(h(x)\big)^3 = (1 + x^2)^3$.
    - So $\frac{\mathrm{d}f(x)}{dx} = \frac{dg\big(h(x)\big)}{dh(x)}\frac{dh(x)}{dx} = 3\big(h(x)\big)^2 2x = 3(1 + x^2)^2(2x)$.

Rules for Calculating Derivatives

Find the derivative $\mathrm{d}f(x)/dx$ of the following functions:

1) $f(x) = -13$

2) $f(x) = 3x^2$

3) $f(x) = 3x^2 - 13$

4) $f(x) = (3x^2 - 13)^3$

5) $f(x) = \frac{3x^2 - 13}{x}$

Finding Maxima and Minima

- Often need to find maxima or minima of functions in microeconomics (utility, cost etc.).
- Derivatives can be used for this.
- Idea: slope of a function at a maximum or minimum of a function is zero.

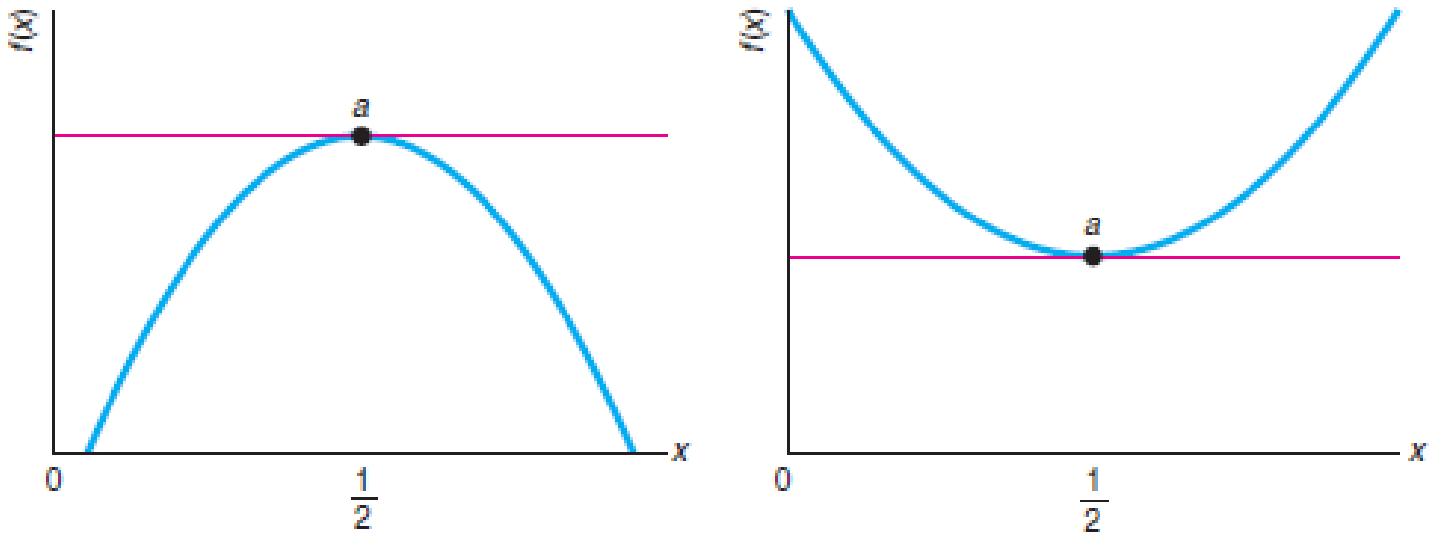


Figure A.7 from Perloff (2014)

Finding Maxima and Minima

- So at a maximum or minimum, $\mathrm{d}f(x)/dx = 0$.
  - Called a "first-order condition".
  - Compute $\mathrm{d}f(x)/dx$ and solve for x ("critical" or "stationary" point).
- Example: find the minimum of $f(x) = x^2 + 1$.
  - $\mathrm{d}f(x)/dx = 2x$
  - Set $2x = 0$, so minimum at $x = 0$.
- Caveats:
  - $\mathrm{d}f(x)/dx = 0$ only necessary condition.
  - Only works for interior maxima and minima.
  - Details in Mathematical/Quantitative Economics module.

Functions of Several Variables

- Will often work with functions of two or more variables: $f(x, y)$, $f(x, y, z)$ etc.
- Can compute ***partial derivatives*** by differentiating with respect to one variable while holding the other variables constant (i.e., treat it as a constant).
- Notation: $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$.
- Example: $f(x, y) = x^2 + y$. Then, $\frac{\partial f(x,y)}{\partial x} = 2x$, $\frac{\partial f(x,y)}{\partial y} = 1$.

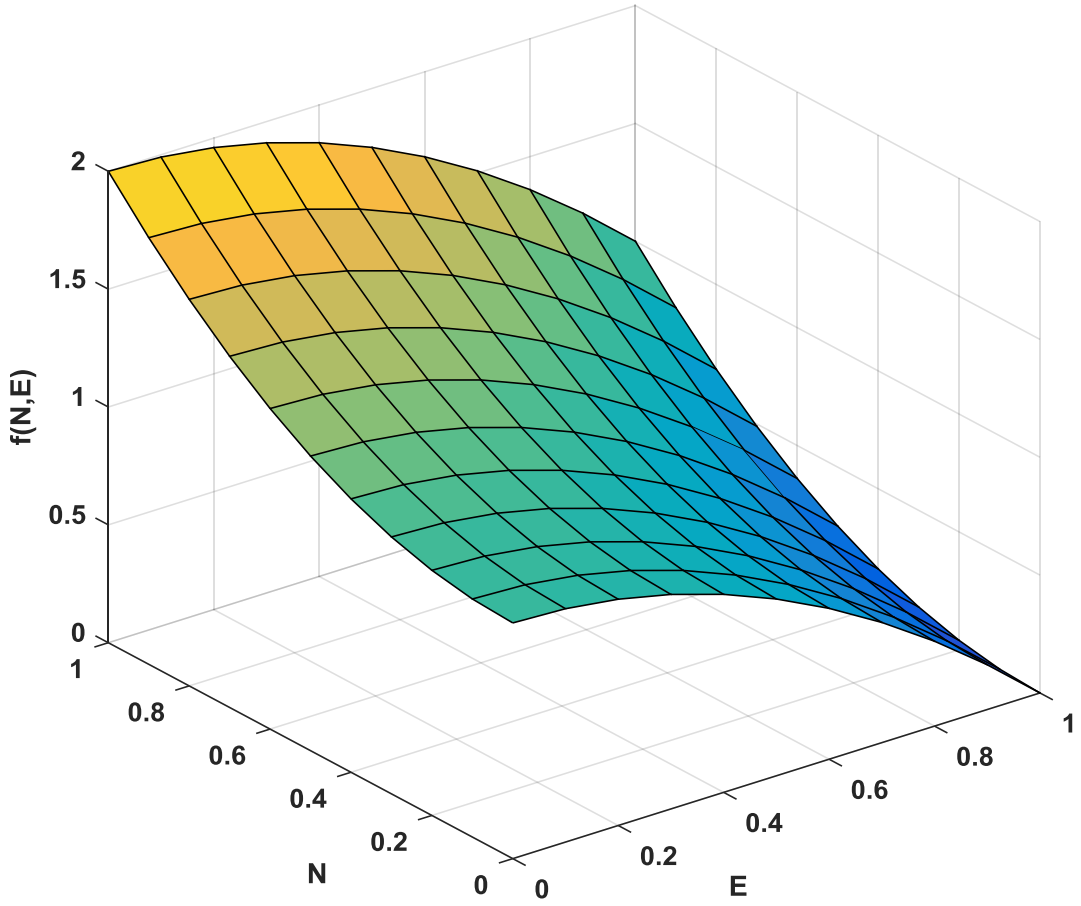
Functions of Several Variables

- Graphical interpretation of partial derivatives: slope of the function $f(x, y)$ in one particular direction.
- Example: contour map where elevation $f(N, E)$ depends on North-East coordinates:

$$f(N, E) = N^2 - E^2 + 1$$

- Partial derivative: how does elevation change if we only move north, or only move east?

Functions of Several Variables



$$f(N,E) = N^2 - E^2 + 1$$

$$\frac{\partial f(N,E)}{\partial N} = 2N$$

$$\frac{\partial f(N,E)}{\partial E} = -2E$$

Functions of Several Variables

- Can again use derivatives to find maxima/minima of functions of several variables.
- Compute partial derivatives with respect to each variable.
  - As many first-order conditions as variables.
  - Example: $f(x, y) = x + x^2 + 2y^2$
    - First-order conditions: $\frac{\partial f(x,y)}{\partial x} = 1 + 2x$, $\frac{\partial f(x,y)}{\partial y} = 4$y
    - Solve for $1 + 2x \Leftrightarrow x = -0.5$ and $4y = 0 \Leftrightarrow y = 0$.
- (Same caveats as before.)

Functions of Several Variables

Find $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$ for the following functions:

1) $f(x,y) = 2x^3 - 11y$

2) $f(x,y) = 2x^3 - 11x^2y + 3y^2$

3) $f(x,y) = (2x+3)(y-2)$

4) $f(x,y) = x^2y^2$

Total Differential

- By how much does a function's value change if we simultaneously change all variables by a small amount?
- Called the ***total differential***:

$$df(x,y) = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy$$

  - (Marginal) change in the value of $f(x,y)$ if we move x by dx and y by dy.
  - Example:

$$f(x,y) = x^2 + y^2,\ df(x,y) = 2xdx + 2ydy$$

Summary & Learning Outcomes

- Brief summary of derivatives of functions of one variable.
- Economists often use functions of several variables, so we looked at the extension to derivatives of functions of two or more variables.
- Will use derivatives throughout the module to illustrate economic concepts. But economic intuition will always be more important!

Reading

- Perloff: Calculus Appendix A.1, A.3 – A.5