Mathematical Economics

Economics Tutors in London

Mathematical Finance
Math for Economists
Quantitative Finance
Calculus for Economists
Financial Derivatives€
Computational Finance
Math of Finance

The derivatives of exponential and logarithmic functions and the fate of Mark's £1471

12.11.19

Previously on Keeping Up With Mark

Mark wants to deposit £1417.

Previously on Keeping Up With Mark

RBS offer him 0.5% per year.

Previously on Keeping Up With Mark

How long will it take him to double the money?

The case of continuous compounding

Assume the interest is added continuously (it's the best possible scenario for him).

$2834 = 1417e^{0.005x} \Leftrightarrow 2 = e^{0.005x}$.

Now it's time to take logs on both sides.

- As before we can pick any base
- This time e is a great choice as it simplifies things

$\ln 2 = \ln e^{0.005x} = 0.005x \ln e = 0.005x \quad \Leftrightarrow$

$x = \frac{ln 2}{0.005} = \frac{0.6931}{0.005} = 138.62$

Ouch!

The case of discrete compounding

Suppose the interest is added only once per year:

$2834 = 1417(1 + 0.005)^x \Leftrightarrow 2 = (1.005)^x$

Let's take logs on both sides (and use the natural log as in the previous case)

$\ln 2 = \ln 1.005^x = x \ln 1.005 \quad \Leftrightarrow \quad x = \frac{\ln 2}{\ln 1.005} = 138.98$

The general case

In fact, we didn't need to know the exact sum of money Mark had.

In case of doubling $2a = a(1+r)^x$ or $2a = ae^{rx}$.

You end up with either

- $\ln 2 = rx$ or
- $\ln 2 = x \ln(1+r)$

and solve for x.

The solution is:

1. Continuous case: $x = \frac{\ln 2}{r}$
2. Discrete case: $x = \frac{\ln 2}{ln(1+r)} \approx \frac{\ln 2}{r}$, because when r is small $ln(1+r) \approx r$

The rule of 69

$\ln 2 = 0.6931 \Rightarrow$ the answer is

- $x \approx \frac{0.69}{r}$, where r is a proportion
- or $x \approx \frac{69}{r}$ where r is a percentage

This is often called **the rule of 69**.

There's one tiny problem: 69 has very few factors (1, 3, 23, 69) so it's hard to divide in your head.

The rule of 72

72 is close enough to 69 and has 12 factors (1,2,3,4,6,8,9,12,18,24,36,72)

The rule of 72

The doubling period of x is approximately $\frac{72}{r}$, where r is the growth rate in percent

69 vs 72

Rate	Actual	69	Error 69	72	Error 72
1%	69.66072	69	0.95%	72	3.36%
2%	35.00279	34.5	1.44%	36	2.85%
3%	**23.44977**	**23**	**1.92%**	**24**	**2.35%**
4%	**17.67299**	**17.25**	**2.39%**	**18**	**1.85%**
5%	14.2067	13.8	2.86%	14.4	1.36%
6%	11.89566	11.5	3.33%	12	0.88%
7%	10.24477	9.857143	3.78%	10.28571	0.40%
8%	9.006468	8.625	4.24%	9	0.07%
9%	8.043232	7.666667	4.68%	8	0.54%
10%	7.272541	6.9	5.12%	7.2	1.00%
11%	6.641885	6.272727	5.56%	6.545455	1.45%
12%	6.116255	5.75	5.99%	6	1.90%
13%	5.671417	5.307692	6.41%	5.538462	2.34%
14%	5.290059	4.928571	6.83%	5.142857	2.78%
15%	4.959484	4.6	7.25%	4.8	3.22%
20%	3.801784	3.45	9.25%	3.6	5.31%
50%	1.709511	1.38	19.28%	1.44	15.77%
70%	1.306276	0.985714	24.54%	1.028571	21.26%

69 vs 72

- 69 is technically the right number to use in continuous cases
- 72 is more convenient
- 72 actually makes up for the error in $ln(1 + r) \approx r$

The derivative of e^x

The derivative of the natural exponential function

Given $y = f(x) = e^x$

$$\frac{dy}{dx} = f'(x) = e^x$$

- e^x is the only function for which the derivative is equal to the value of the function at any point

If we apply the chain rule we can generalise the rule:
Given $y = f(g(x)) = e^{g(x)}$

$$\frac{dy}{dx} = e^{g(x)} \cdot g'(x)$$

The derivative of $\ln x$

The derivative of the natural logarithmic function

Given $y = f(x) = \ln x$

$$\frac{dy}{dx} = f'(x) = \frac{1}{x}$$

- The slope of the natural logarithm is inversely proportional to the value of x

If we apply the chain rule we can generalise the rule:

Given $y = f(g(x)) = \ln(g(x))$

$$\frac{dy}{dx} = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$$

Examples

Example 1

Consider the case of continuous growth $y = ae^{rx}$. By how much would the value of y go up in the next yoctosecond?

Using notation from the previous slides $y = f(g(x)) = ae^{g(x)}$, where $g(x) = rx$.

$$\frac{dy}{dx} = \underbrace{a}_{\text{multiplicative constant}} \cdot \overbrace{e^{rx}}^{\text{the derivative of the exponent}} \cdot \underbrace{r}_{\text{the derivative of } g(x) = rx}$$

Examples

Example 2

Given $y = \ln(x^2 + 3x)$ find $\frac{dy}{dx}$

Simply apply the chain rule and use the expression for the derivative of the logarithm. Or apply $\ln(g(x))' = \frac{g(x)'}{g(x)}$ directly:

$$\frac{dy}{dx} = \underbrace{\frac{1}{x^2 + 3x}}_{\text{The derivative of } \ln} \cdot \overbrace{(2x + 3)}^{\text{The derivative of } x^2 + 3x}$$

Growth rate

$y = f(x)$

- a change Δx results in Δy
- $\frac{\Delta y}{\Delta x}$ – the change in y per unit of x
- the **proportionate** change in y per unit change in x is

$$\frac{\frac{\Delta y}{y}}{\Delta x} = \frac{1}{y}\frac{\Delta y}{\Delta x}$$

What if growth isn't constant

Year	CPI
2010	100
2011	101
2012	102
2013	104
2014	107
2015	110

The growth rate over 1 period: 2010 - 2011: $\frac{1}{y}\frac{\Delta y}{\Delta x} = \frac{1}{100}\frac{1}{1} = 1\%$ ✓
The growth rate over 5 periods: $\frac{1}{100}\frac{10}{5} = 2\%$ **X**

Use either $y = a(1+r)^x$ or $y = ae^{rx}$. This assumes constant proportional rather than absolute growth.

$$110 = 100e^{5r} \Leftrightarrow r = \frac{\ln 1.1}{5} = 1.9\%$$

Instantaneous growth

Discrete growth rate:

$$\frac{1}{y}\frac{\Delta y}{\Delta x}$$

Instantaneous growth rate:

$$\lim_{\Delta x \to 0} \frac{1}{y}\frac{\Delta y}{\Delta x} = \frac{1}{y}\frac{dy}{dx}$$

Instantaneous growth: example

Example 3

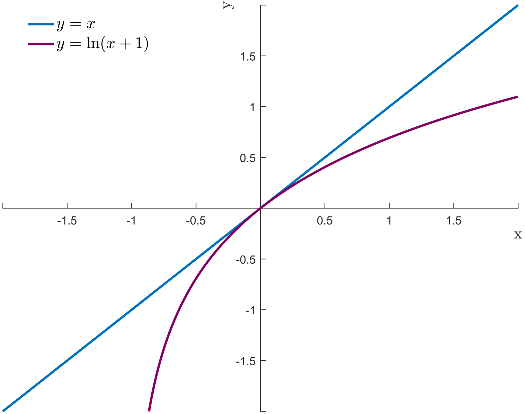
$y = ae^{rx}$ what is the instantaneous growth rate?

$$\frac{1}{y} = \frac{1}{ae^{rx}}$$

$$\frac{dy}{dx} = ae^{rx} \cdot r \Rightarrow$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{ae^{rx}r}{ae^{rx}} = r$$

Logarithms as proportional changes



$\ln(1+x) \approx x$ around 0

Logarithms as proportional changes

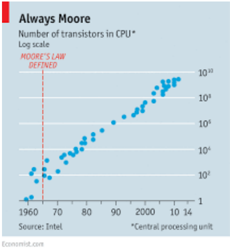
$$\ln\left(1+\frac{\Delta y}{y}\right) \approx \frac{\Delta y}{y}$$

$$\ln\left(\frac{y}{y}+\frac{\Delta y}{y}\right) \approx \frac{\Delta y}{y}$$

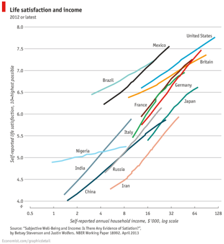
$$\ln\left(\frac{\Delta y + y}{y}\right) \approx \frac{\Delta y}{y}$$

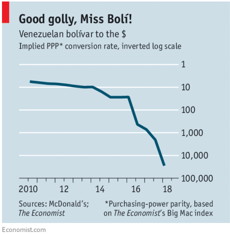
$$\ln(y+\Delta y) - lny \approx \frac{\Delta y}{y}$$

The change in the log is approximately equal to the proportionate change in y


Always Moore
Number of transistors in CPU*
Log scale
MOORE'S LAW DEFINED
1
10²
10⁴
10⁶
10⁸
10¹⁰
1960
70
80
90
2000
10
14
Source: Intel
*Central processing unit
Economist.com


Coining it
$ per bitcoin
Log scale
10,000
1,000
100
10
1
0.1
0.01
2010
11
12
13
14
15
16
17
Source: CoinDesk
Economist.com


Life satisfaction and income
2012 or latest
Self-reported life satisfaction, 10=highest possible
8.0
7.5
7.0
6.5
6.0
5.5
5.0
4.5
4.0
United States
Mexico
Britain
Brazil
Germany
Japan
France
Italy
Nigeria
India
Russia
Iran
China
0.5
1
2
4
8
16
32
64
128
Self-reported annual household income, $'000, log scale
Source: "Subjective Well-Being and Income: Is There Any Evidence of Satiation?", by Betsey Stevenson and Justin Wolfers. NBER Working Paper 18992. April 2013
Economist.com/graphicdetail


Good golly, Miss Bolí!
Venezuelan bolívar to the $
Implied PPP* conversion rate, inverted log scale
1
10
100
1,000
10,000
100,000
2010
12
14
16
18
Sources: McDonald's; The Economist
*Purchasing-power parity, based on The Economist's Big Mac index
Economist.com

Semi-log scale and growth rate

Consider $y = f(x)$. Often you see graphs where x is on the horizontal axis and $\ln y$ is on the vertical axis.

- The most common example: y is GPD and x is time

Why do we do it? Let's start with an example.

One reason why we like logs

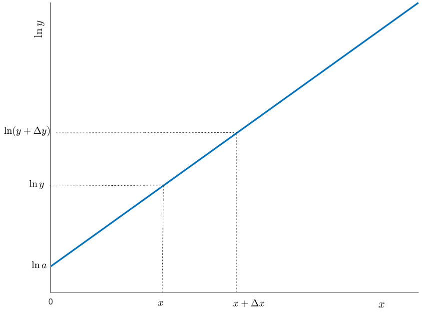
Take a variable that grows continuously at constant rate

$$y = ae^{rx} \Rightarrow \ln y = \ln(ae^{rx}) \Leftrightarrow$$

$$\ln y = \ln a + rx$$

- We have a linear relationship between $\ln y$ and x
- If you see an ugly highly non-linear functional form – take logs. It will make your life much easier

Back to growth rates



- The slope: $r = \frac{\ln(y+\Delta y)-\ln(y)}{x+\Delta x-x} = \frac{\Delta y}{y}\frac{1}{\Delta x} = \frac{1}{y}\frac{\Delta y}{\Delta x}$ or
- $\ln y = \ln a + rx \Rightarrow \frac{d\ln y}{dx} = r$
- In general: $y = f(x) \Rightarrow$ using the chain rule $\frac{d\ln y}{dx} = \frac{1}{y}\frac{dy}{dx}$
- The slope of a semi-log graph is the instantaneous growth rate

Log-log scale

Say you have a function $y = f(x)$. If you take logs on both sides, $\ln y = \ln f(x)$, and plot it:

- The slope of the graph is the elasticity
- It's the rate of proportionate change in y per unit of proportionate change in x

Why? Consider an example:

Elasticity of demand

Suppose you estimate the demand function for some product: $q = Ap^{-\alpha}$.

The elasticity of demand is given by:

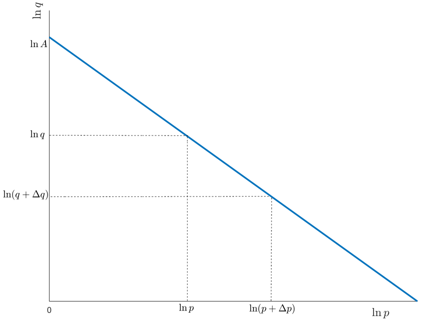
$$E^D = \frac{p}{q}\frac{dq}{dp} = \frac{p}{Ap^{-\alpha}} \cdot -\alpha A p^{-\alpha-1} = \frac{p}{Ap^{-\alpha}}\frac{Ap^{-\alpha}}{p} \cdot -\alpha = -\alpha$$

Take logs on both sides:

$$\ln q = \ln A - \alpha \ln p$$

https://econtutor.com/london-universities/econometrics-tutor-london-universities/

Elasticity of demand



We have a linear relationship once again. Really, it's a very powerful trick – use it!

The slope is $-\alpha$:

- $\ln q = \ln A - \alpha \ln p \Rightarrow \frac{d \ln q}{d \ln p} = -\alpha$
- $-\alpha = \frac{\ln(q+\Delta q) - \ln(q)}{\ln(p+\Delta p) - \ln p} = \frac{\Delta q}{q} \frac{p}{\Delta p} = \frac{p}{q} \frac{\Delta q}{\Delta p}$

Elasticities and log-log scale

I used an example to show that the slope of a log-log graph is the elasticity.

- But it is a general result
- Works with multiple variables e.g. $z = f(x, y) \Rightarrow$ the partial elasticities are:

1. $$\frac{\partial \ln z}{\partial \ln x} = \frac{x}{z}\frac{\partial z}{\partial x}$$

2. $$\frac{\partial \ln z}{\partial \ln y} = \frac{y}{z}\frac{\partial z}{\partial y}$$

So, the logs

- At first look scary, but only at first (a great Halloween costume!)
- Make things nicer-looking (i.e. linear)
- Can be interpreted as elasticities and semi-elasticities (handy)
- Make it easier to visualise growth rates

Some nice properties of growth rates

The growth rate of a product and a ratio

1. The growth rate of a product is the sum of the growth rates
2. The growth rate of a ratio is the difference between the growth rates
3. The growth rate of x^α is ($\alpha \cdot$ the growth rate of x)

Consider $z_1 = x \cdot y$, $z_2 = \frac{x}{y}$, and $z_3 = x^\alpha$.

Suppose x and y grow continuously at constant rates g_x and g_y respectively.

The logs will help to see it's true

The growth rate of z_1: $\frac{1}{z_1}\frac{dz_1}{d(xy)} = \ln(z_1)'$

To find the growth rate we need to take the log and find the derivative.

Using the properties of the logarithmic function we can write:
$\ln z_1 = \ln(xy) = \ln x + \ln y \Rightarrow$

$$\ln(z_1)' = (\ln x + \ln y)' = \ln(x)' + \ln(y)' = g_x + g_y$$

The growth rate of $z_2 = \frac{1}{z_2}\frac{dz_2}{d\left(\frac{x}{y}\right)} = \ln(z_2)'$

$$\ln(z_2)' = ln\left(\frac{x}{y}\right)' = (\ln x - \ln y)' = \ln(x)' - \ln(y)' = g_x - g_y$$

The growth rate of $z_3 = \ln(z_3)' = \ln(x^{\alpha})' = \alpha \ln(x)' = \alpha g_x$