Integration

Maths Tutors London

Mathematical Finance
Math for Economists
Quantitative Finance
Calculus for Economists
Financial Derivatives
Computational Finance
Math of Finance

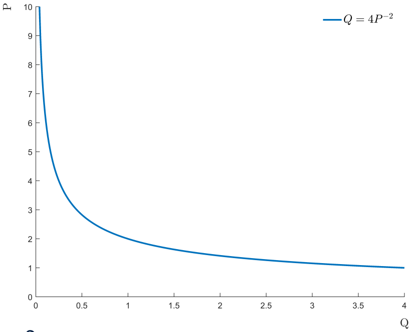
Essentials of Integration

19.11.19

You've learnt a lot and may feel

But what if...

The demand (and supply) functions are non-linear?



$Q_d = 4P^{-2} \Leftrightarrow P = \frac{2}{\sqrt[2]{Q_d}}$; This function has a useful property. What is it?
(see week 6)

But we can no longer use the formula for the area of a triangle to find the surpluses :(

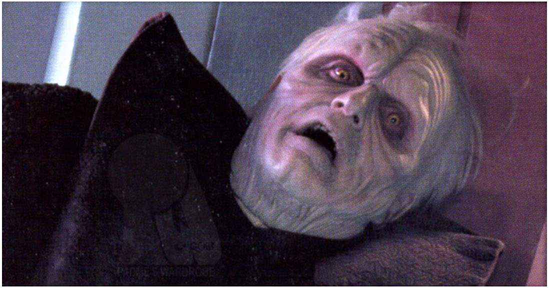
But what if...

But what if you need to calculate expectation of a continuous random variable?

- For example you are considering buying an asset linked to a financial index
- You already know that expectation of a discrete variable is $\mathbb{E}(X) = \sum_{i=1}^{n} P_i X_i$, where P_i is the probability of X taking the value X_i

Suppose you also care about the future when you maximise the profit of your firm (or your utility).

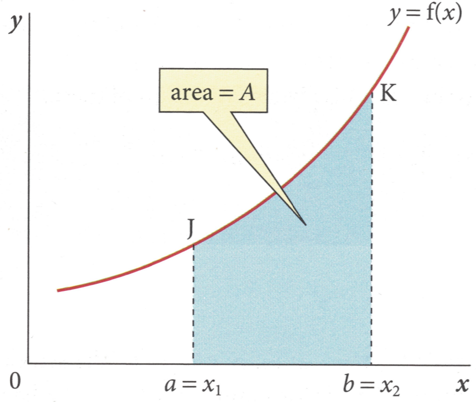
- $\pi = \pi_1 + \beta\pi_2 + \ldots + \beta\pi_t + \ldots$, where β is a discount factor
- But what if instead of working with *periods* ("today", "tomorrow" etc) you want to work in continuous time



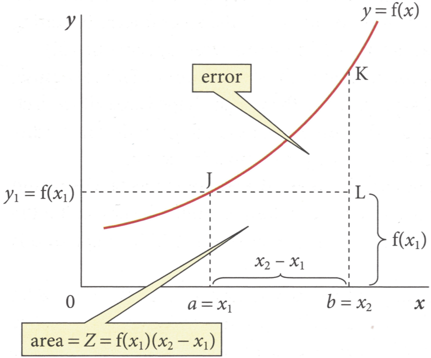
Don't worry, there are maths tools which will save the day

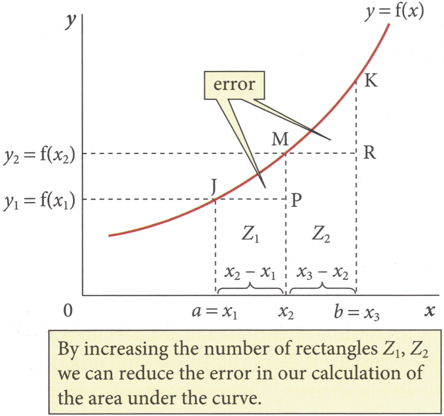
Measuring the area under a curve

Consider a function $y = f(x)$. Suppose we are interested in measuring the area A under that function and between two values of x: a and b.



Measuring the area under a curve





We can approximate the area by a rectangle. Or several of them. The more we use, the smaller the error.

Approximation of the area under the curve

If we use only one rectangle: $A \approx Z = \underbrace{f(x_1)}_{height}\underbrace{(x_2 - x_1)}_{width}$

If we use two rectangles: $A \approx Z_1 + Z_2 = f(x_1)(x_2 - x_1) + f(x_2)(x_3 - x_2)$

If we keep adding more and more rectangles: $A \approx \sum_{i=1}^{n} f(x_i)(x_{i+1} - x_i)$

When the number of rectangles becomes infinitely large, the sum of their areas approaches the shaded area A:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)(x_{i+1} - x_i)$$

The definite integral

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)(x_{i+1} - x_i) = \int_a^b f(x)dx$$

- As you already know, dx is an infinitely small change in x
- $\int$ is a stylised letter S
- a and b are called the *limits of integration*
- The expression reads: *"A is the definite integral of the function $f(x)$ between the limits a and b"*

The indefinite integral

If the limits of integration are not specified the integral measures an open-ended area and is called the *indefinite integral*.

$$\int f(x)dx$$

Now we know the notation. But how do we evaluate integrals?

The fundamental theorem of calculus

Theorem 1 (The first fundamental theorem of calculus)

$\int f(x)dx = F(x) + c$, *where* $\frac{dF(x)}{dx} = f(x)$ *and* c *is an arbitrary constant.*

In English, integration is the reverse of differentiation

Theorem 2 (The second fundamental theorem of calculus)

If $\int f(x)dx = F(x) + c$, *then* $\int_a^b f(x)dx = F(x = b) - F(x = a)$

We won't prove these theorems, but here's an intuition that may help to understand them.

Physical intuition: the first theorem

Imagine you are driving a car but the odometer is broken.

Therefore, you have no idea how far you've travelled.

However your speedometer works and you have a stopwatch.

- You can check the speed and measure an interval of time
- Multiplying the two will tell you how far you travelled in that interval of time

Now imagine doing it time after time, so that at any point in time you know how far you travelled during the previous interval.

It's an approximation of the total distance you've covered.

Physical intuition: the first theorem

However, it's hard to maintain a constant speed.

For the approximation to be any good you need to make the time intervals very small.

If you make them infinitely small you'll know the exact distance.

Physical intuition: the first theorem

The total distance is the integral of your speed.

At the same time, speed is the first derivative of covered distance. It tells you how the distance changes.

Physical intuition: the second theorem

Now think about how our travelling example would apply to the definite integral.

Imagine you fixed your odometer and decided to go for a ride between 11am and 3pm.

If you decided to check your speed and multiply it by the periods of time only between 1pm and 2pm, you would get an approximation of the distance covered between 1pm and 2pm.

The definite integral of speed between 1pm and 2pm would give you the exact distance covered in that time period.

But the difference between the odometer readings at 2pm and 1pm gives you exactly the same information.

The basic rules of integration

1. Power rule of integration: $\int x^n dx = \frac{x^{n+1}}{n+1} + c$, provided $n \neq -1$
2. Multiplicative constants: $\int Af(x)dx = A\int f(x)dx = AF(x) + c$,
3. Sums or differences:
   $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx = F(x) + G(x) + c$
4. Exponential function: $\int e^x dx = e^x + c$
   $\int f'(x)e^{f(x)}dx = e^{f(x)} + c$
5. Logarithmic function: $\int \frac{1}{x}dx = ln(x) + c$
   $\int \frac{f'(x)}{f(x)}dx = ln(f(x)) + c$
6. Function of a function (the substitution rule):
   $\int f(u)\frac{du}{dx}dx = \int f(u)du = F(u) + c$

Integration by parts

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

Why is it the case?

The product rule of differentiation says $\frac{d(u(x)v(x))}{dx} = u'(x)v(x) + u(x)v'(x)$

Integrate both sides to get $u(x)v(x) = \int u'(x)v(x)dx + \int u(x)v'(x)dx$, which can be rearranged to obtain the expression above.

Integration by parts

It may not seem like a useful formula, but it actually is.

(Isn't it the case with all superheroes??)

Examples

Example 1

Find $\int x^3 dx$

Using the power rule we get $\int x^3 dx = \frac{x^4}{4} + c$.

Example 2

Find $\int x^{0.5} dx$

Using the power rule we get $\int x^{0.5} dx = \frac{x^{1.5}}{1.5} + c$.

Examples

Example 3

Find $\int 100x^3 dx$

100 is a multiplicative constant, therefore

$\int 100x^3 dx = 100 \int x^3 dx = 100\frac{x^4}{4} + c = 25x^4 + c$

Example 4

Find $\int 2xe^{x^2} dx$

The first derivative of x^2 is $2x$. Therefore, $2xe^{x^2} = f'(x)e^{f(x)}$, where $f(x) = x^2$.

Thus, using the exponential function rule we get $\int 2xe^{x^2} dx = e^{x^2} + c$.

Examples

Example 5

Find $\int \frac{3x^2}{x^3+1} dx$

The first derivative of $x^3 + 1$ is $3x^2$. Therefore, using the logarithmic function rule we get $\int \frac{3x^2}{x^3+1} dx = ln(x^3 + 1) + c$.

Example 6

Find $\int (3x^2 + 3)(x^3 + 3x) dx$

Let's denote $u = x^3 + 3x$. Then $\frac{du}{dx} = 3x^2 + 3$, It follows that $du = (3x^2 + 3)dx$.

Now consider the integral
$\int (3x^2 + 3)(x^3 + 3x) dx = \int u du = \frac{u^2}{2} + c = \frac{(x^3+3x)^2}{2} + c$.

Examples

Example 7

Find $\int ln(x)dx$

Here's where integration by parts will save us.

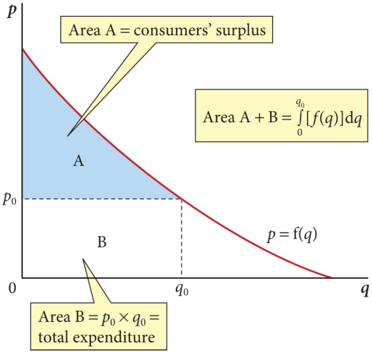
$\int ln(x)dx = \int 1 \cdot ln(x)dx.$

Let's denote $u(x) = ln(x)$ and $v(x) = x$. Then $u'(x) = \frac{1}{x}$ and $v'(x) = 1$.

Now integrate $\int ln(x)dx$ by parts:

$$\int \underbrace{1}_{v'(x)} \cdot \underbrace{ln(x)}_{u(x)} dx = xln(x) - \int \frac{1}{x} x dx = xln(x) - \int 1 \cdot dx = xln(x) - x + c$$

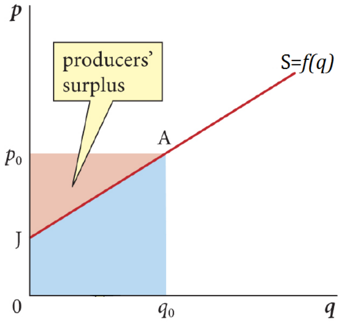
Consumer surplus



$$CS = \left(\int_0^{q_0} f(q)dq\right) - p_0 q_0$$

Producer surplus

A glimpse of Economics 2



$$PS = p_0 q_0 - \int_0^{q_0} f(q)dq$$

Example: the consumer surplus

Example 8

The demand function is estimated to be $Q_d = 64P^{-2} - 2$ Find the consumer surplus if the market price is $P_0 = 4$.

To find the surplus we will be integrating over quantity. Therefore, let's find the inverse functions and the equilibrium quantity.

Example: the consumer surplus

Demand: $Q_d = 64P^{-2} - 2 \quad \Leftrightarrow \quad 64P^{-2} = Q_d + 2 \quad \Leftrightarrow \quad P = \frac{8}{\sqrt{Q+2}}$

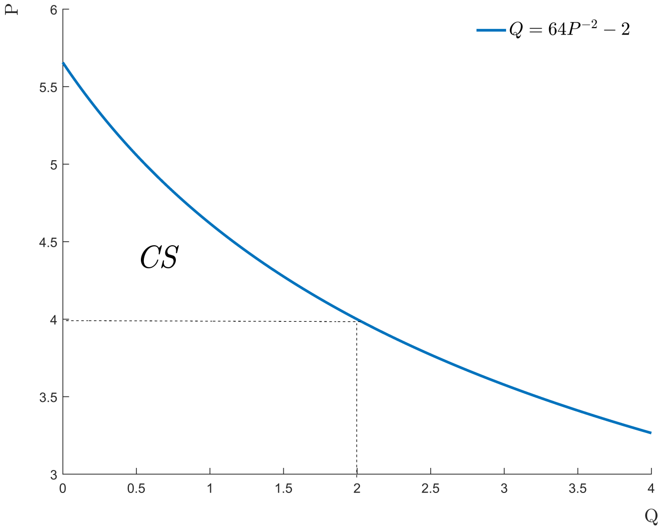
$\frac{8}{\sqrt{Q+2}} = 4 \Rightarrow Q_0 = 2$

The consumer surplus:

$CS = \int_0^2 \frac{8}{\sqrt{Q+2}} dQ - P_0 Q_0 = 8 \int_0^2 (Q+2)^{-0.5} dQ - 4 \cdot 2 =$

$16(Q+2)^{0.5}\Big|_0^2 - 8 = 16 \cdot 2 - 16 \cdot \sqrt{2} - 8 \approx 1.37$

The graph



Example: the definite integral (by parts)

Example 9

Find $\int_1^2 xe^{6x}dx$

Integration by parts may be quite handy

$\int_a^b u(x)v'(x)dx = u(x)v(x)|_a^b - \int_a^b u'(x)v(x)dx$

Let $u(x) = x$ and $v'(x) = e^{6x}$.

Then $u'(x) = 1$ and $v(x) = \int e^{6x}dx$.

We can use the exponential function rule to find $v(x)$:

$v(x) = \int \frac{1}{6}6e^{6x}dx = \frac{1}{6}\int 6e^{6x}dx = \frac{1}{6}e^{6x} + c.$

Now we can write:

$$\int_1^2 xe^{6x}dx = x \cdot \frac{1}{6}e^{6x}|_1^2 - \frac{1}{6}\int_1^2 e^{6x}dx = \frac{x}{6}e^{6x}|_1^2 - \frac{1}{6}\cdot\frac{1}{6}\int_1^2 6e^{6x}dx =$$

$$= \frac{x}{6}e^{6x}|_1^2 - \frac{1}{36}e^{6x}|_1^2 = \left(\frac{2}{6}e^{6\cdot 2} - \frac{1}{6}e^{6\cdot 1}\right) - \left(\frac{1}{36}e^{6\cdot 2} - \frac{1}{36}e^{6\cdot 1}\right) =$$

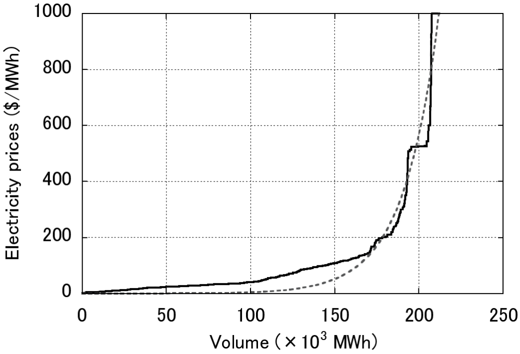
$$= \frac{11}{36}e^{12} - \frac{5}{36}e^{6} = \frac{e^6}{36}(11e^6 - 5)$$

A bit of Yoda's wisdom and foresight

More useful than you think it is.

- Increasing and convex the function $f(x) = xe^{6x}$ is.
  DIY: check it
- A supply function it could be
- Exponential supply? Unreasonable you say?

Supply of electricity



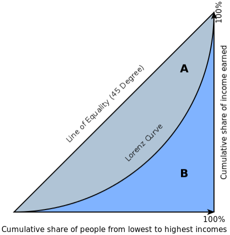
Solid line: PJM Interconnection (USA) supply function

Dashed line: exponential function (the general form: $P = \alpha(e^{\beta Q} - 1)$)

[Takashima et al., 2007]

Some other applications

- Moments of random variables, e.g. expectation $E(x) = \int_{-\infty}^{+\infty} xf(x)dx$
- Continuous time optimisation problems, e.g. $U = \int_0^{+\infty} e^{-rt} u(c_t)dt$
- Various aggregators in models with many goods e.g.
  - $C_t = \left(\int_0^1 C_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}$
  - $P_t = \left(\int_0^1 P_t(i)^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}}$
- Income concentration index – the Gini coefficient $G = A/(A+B)$



(fig. from wikipedia)

Takashima, R., Naito, Y., Kimura, H., and Madarame, H. (2007).
Investment in electricity markets: equilibrium price and supply function.
In *11th Annual Real Options Conference, Berkeley, CA, USA, June*, pages 6–9. Citeseer.