Math for Economics

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The exponential function, logarithms and continuous growth

05.11.19

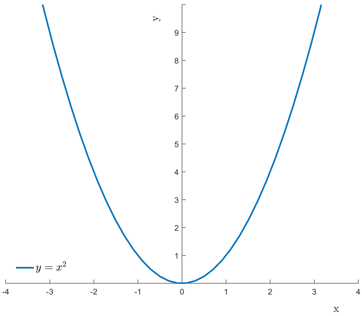
How long to double the money?

Mark, a loyal RBS customer, thinks about depositing £1417

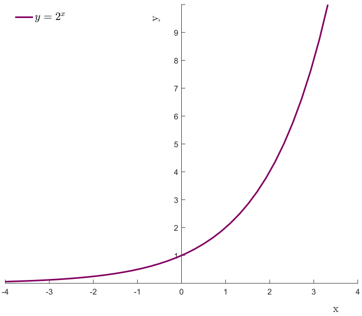
- RBS offer 0.5% per year
  - The interest would be added once a year

How long will it take to double the money?

Exponential vs power functions



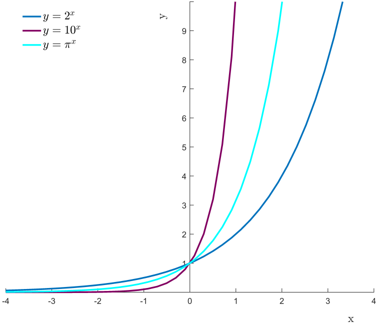
$$y = x^2$$



$$y = 2^x$$

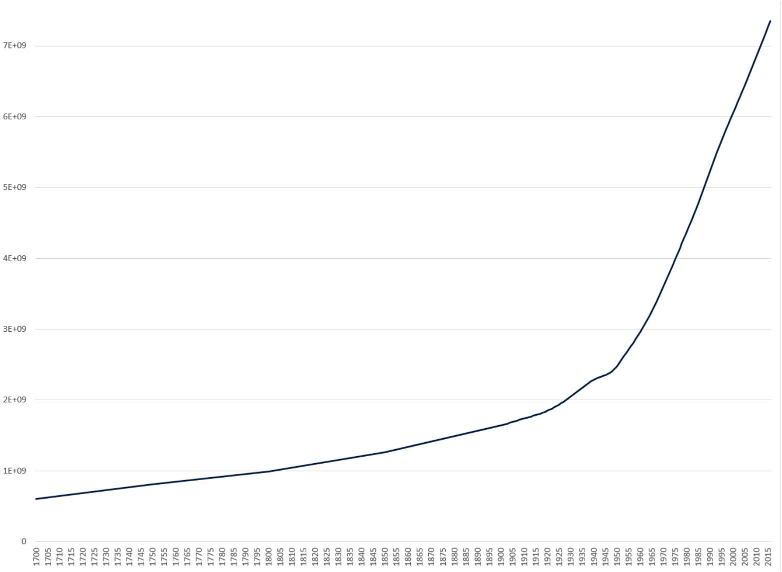
The exponential function

$$y = a^x, \quad a > 0$$

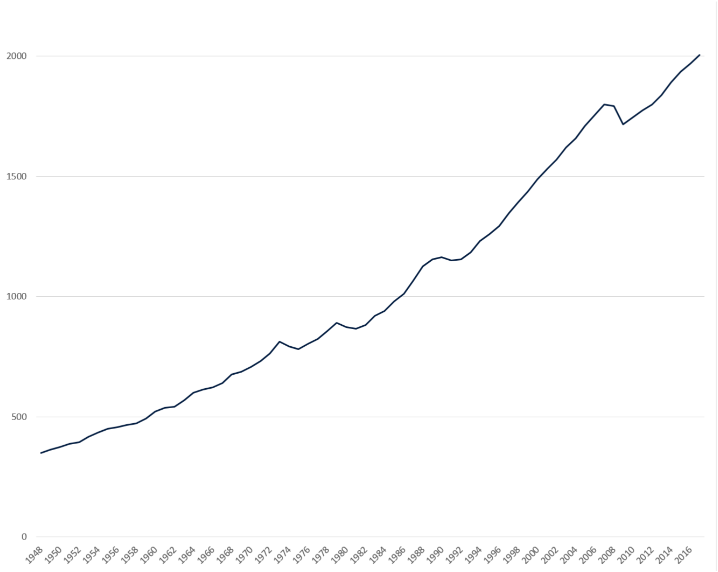


- The function always passes through $(x, y) = (0, 1)$
- y never equals 0
- y grows fast

The world population

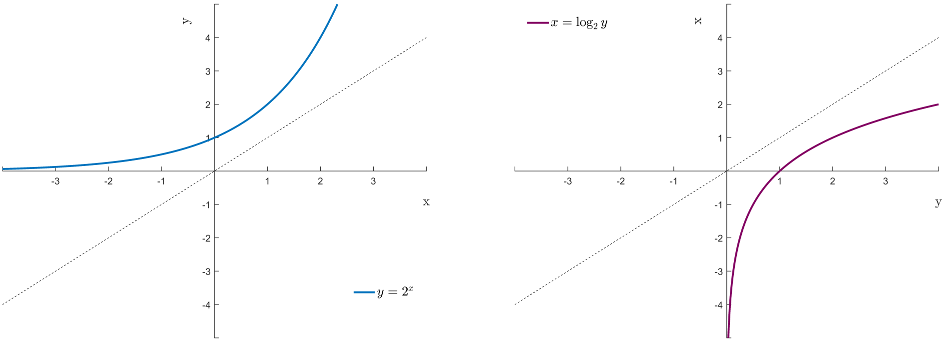


UK's GDP, bil.



2000
1500
1000
500
0
1948 1950 1952 1954 1956 1958 1960 1962 1964 1966 1968 1970 1972 1974 1976 1978 1980 1982 1984 1986 1988 1990 1992 1994 1996 1998 2000 2002 2004 2006 2008 2010 2012 2014 2016

The inverse of the exponential function



The logarithmic function

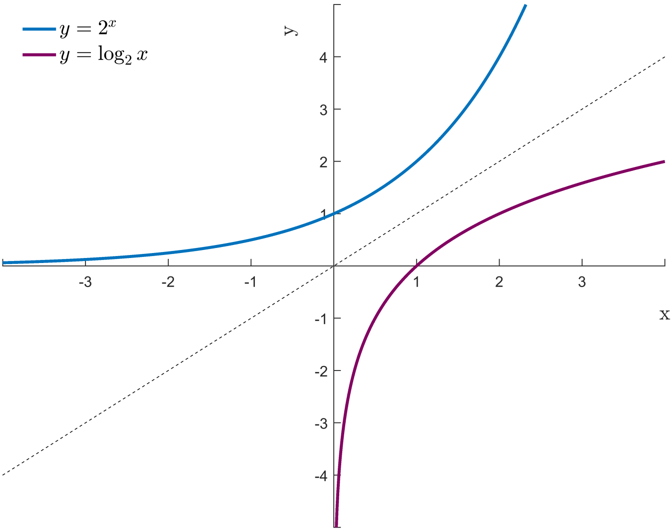
Definition

Given the exponential function $y = a^x$,

- The inverse function is defined as: $x = \log_a y$
- $log_a y$ means "the logarithm to base a of y"

- In some sense logarithms are just another notation for powers
- To what power should I raise a to get y?

Back to normal axes



Some important features

- The curve never cuts the vertical axis
- Real based logarithms are defined for positive numbers only
- The logarithm of $0 < x < 1$ is negative
  - To get a fraction you need to raise 2 to a negative power e.g. $2^{-2} = \frac{1}{2^2}$
  - This is true for any base greater than 1: $\log_a x < 0, \quad a > 1, \quad x < 1$

Rules for Logarithms

Assume A, B and C are positive.

1. $A = B \Leftrightarrow \log_a A = \log_a B$
2. $\log_a(AB) = \log_a A + \log_a B$
3. $\log_a \left(\frac{A}{B}\right) = \log_a A - \log_a B$
4. $\log_a A^n = n \log_a A$
5. $\log_a 1 = 0$
6. $\log_B A = \frac{\log_C A}{\log_C B}$

$log_{10}A \equiv logA$ is known as **the common logarithm**

One useful application

Example 1

$y = a(1+r)^x$, solve for x.

How long does Mark have to wait for his £1417 to become £1500 given $r = 0.005$?

$1500 = 1417(1+0.005)^x$. Divide both sides by 1417:

$\frac{1500}{1417} = 1.005^x$. Pick a base and take the log on both sides.

One useful application

You can pick any base. Good choices include:

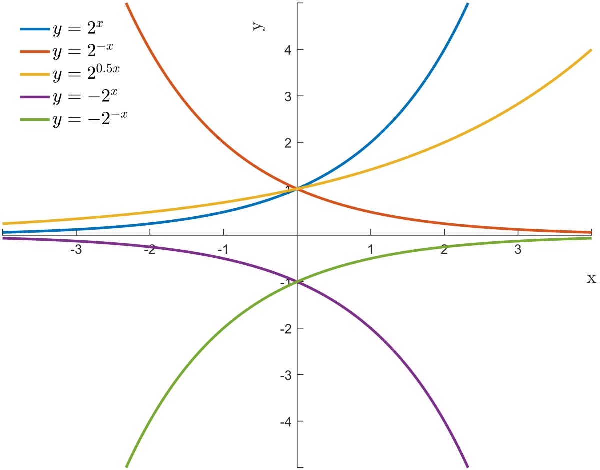
- 10 i.e. the common logarithm (log), you'll always find it on your calculator
- Some number that would turn a part of the equation into 1 e.g. $\frac{1500}{1417}$. In this case it's quite an ugly number

$$\log \frac{1500}{1417} = \log 1.005^x \Leftrightarrow \log \frac{1500}{1417} = x \log 1.005 \Leftrightarrow$$
$$x = \frac{\log \frac{1500}{1417}}{\log 1.005} \approx 11.42$$

He'll have to wait 11.5 years!! It gives you a hint about how long he'll have to wait for his money to double.

A few less standard exponential functions

$$y = a^{bx}$$



Last week: on the way to continuous growth

If the interest is added n times per period:

$$y = a\left(1 + \frac{r}{n}\right)^{nx}$$

Assume $a = 1$, $x = 1$, and $r = 1 = 100\%$

Investment	Formula	Take home
$n = 1$	$y = \left(1 + \frac{1}{1}\right)^{1}$	2
$n = 2$	$y = \left(1 + \frac{1}{2}\right)^{2}$	2.25
$n = 10$	$y = \left(1 + \frac{1}{10}\right)^{10}$	2.5937
$n = 100$	$y = \left(1 + \frac{1}{100}\right)^{100}$	2.7048
$n = 1000000$	$y = \left(1 + \frac{1}{1000000}\right)^{1000000}$	2.7183

e

Euler's number

$$e \equiv \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

- It's irrational just like π, $\sqrt{2}$ i.e. it can't be expressed as a ratio of two integers
- Was discovered by Jacob Bernoulli
  - That was one [crazy] great family, **link**
-

$$e = \sum_{n=0}^{\infty}\frac{1}{n!} = \frac{1}{1}+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\ldots$$

Compound interest in continuous time

The formula

$$\lim_{n\to\infty} a\left(1+\frac{r}{n}\right)^{nx} = ae^{rx}$$

Derivation

$$\lim_{n\to\infty} a\left(1+\frac{r}{n}\right)^{nx} = \lim_{n\to\infty} a\left[\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rx} =$$

$$\overset{w=\frac{n}{r}}{=} \lim_{w\to\infty} a\left[\left(1+\frac{1}{w}\right)^{w}\right]^{rx} = ae^{rx}$$

Discounting in continuous time

Last week we learnt that if discounting takes place n times per period, $y = \frac{a}{\left(1+\frac{r}{n}\right)^{nx}}$. To get the expression in continuous time set $n \to \infty$

The formula in continuous time

$$y = ae^{-rx}$$

Derivation:

$$y = \lim_{n\to\infty} \frac{a}{\left(1+\frac{r}{n}\right)^{nx}} = \frac{a}{e^{rx}} = ae^{-rx}$$

Examples

Example 2

Bhutan's GPD grew by 6% in 2016. Suppose it will keep growing continuously at the same rate for the next 5 years. What will the cumulative percentage growth over these 5 years be?

Take the 2016 level ($2.2 billion) as 100%.

$y = ae^{rx}$, $a = 100$, $r = 0.06$, $x = 5 \Rightarrow$

$y = 100e^{0.3} \approx 135 \Rightarrow$ the percentage growth is $\frac{135-100}{100} = 0.35 = 35\%$

Examples

Example 3

You know you'll have £100 in a year. How much can you borrow at 10% interest if it is added continuously?

$y = ae^{-rx}$, $a = 100$, $r = 0.1$, $x = 1 \Rightarrow$

You can borrow: $y = 100e^{-0.1} \approx 90.48$

The effective growth rate

In case of continuous growth, $y = ae^{rx}$, the effective growth rate is given by

$$EAR = e^r - 1$$

Last week we saw that that for a variable that grows discontinuously

$EAR = \left(1 + \frac{r}{n}\right)^n - 1$

The natural logarithm

The inverse of $y = e^x$

By definition the function inverse to $y = e^x$ is

$$x = \log_e y \equiv \ln y$$

The same standard rules apply:

1. $A = B \Leftrightarrow \ln A = \ln B$
2. $\ln(AB) = \ln A + \ln B$
3. $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$
4. $\ln A^n = n \ln A$
5. $\ln 1 = 0, \ln e = 1$

What about Mark?

We still don't know how long it will take him to double £1417. Next week.