# Multiple Regression Analysis

- $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots \beta_k x_k + u$
- 2. Inference

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# Assumptions of the Classical Linear Model (CLM)

- So far, we know that given the Gauss-Markov assumptions, OLS is BLUE,
- In order to do classical hypothesis testing, we need to add another assumption (beyond the Gauss-Markov assumptions)
- Assume that $u$ is independent of $x_1, x_2,\ldots, x_k$ and $u$ is normally distributed with zero mean and variance $\sigma^2$: $u \sim \text{Normal}(0,\sigma^2)$

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# CLM Assumptions (cont)

- Under CLM, OLS is not only BLUE, but is the minimum variance unbiased estimator
- We can summarize the population assumptions of CLM as follows
- $y|\mathbf{x} \sim \text{Normal}(\beta_0 + \beta_1 x_1 + \ldots + \beta_k x_k, \sigma^2)$
- While for now we just assume normality, clear that sometimes not the case
- Large samples will let us drop normality

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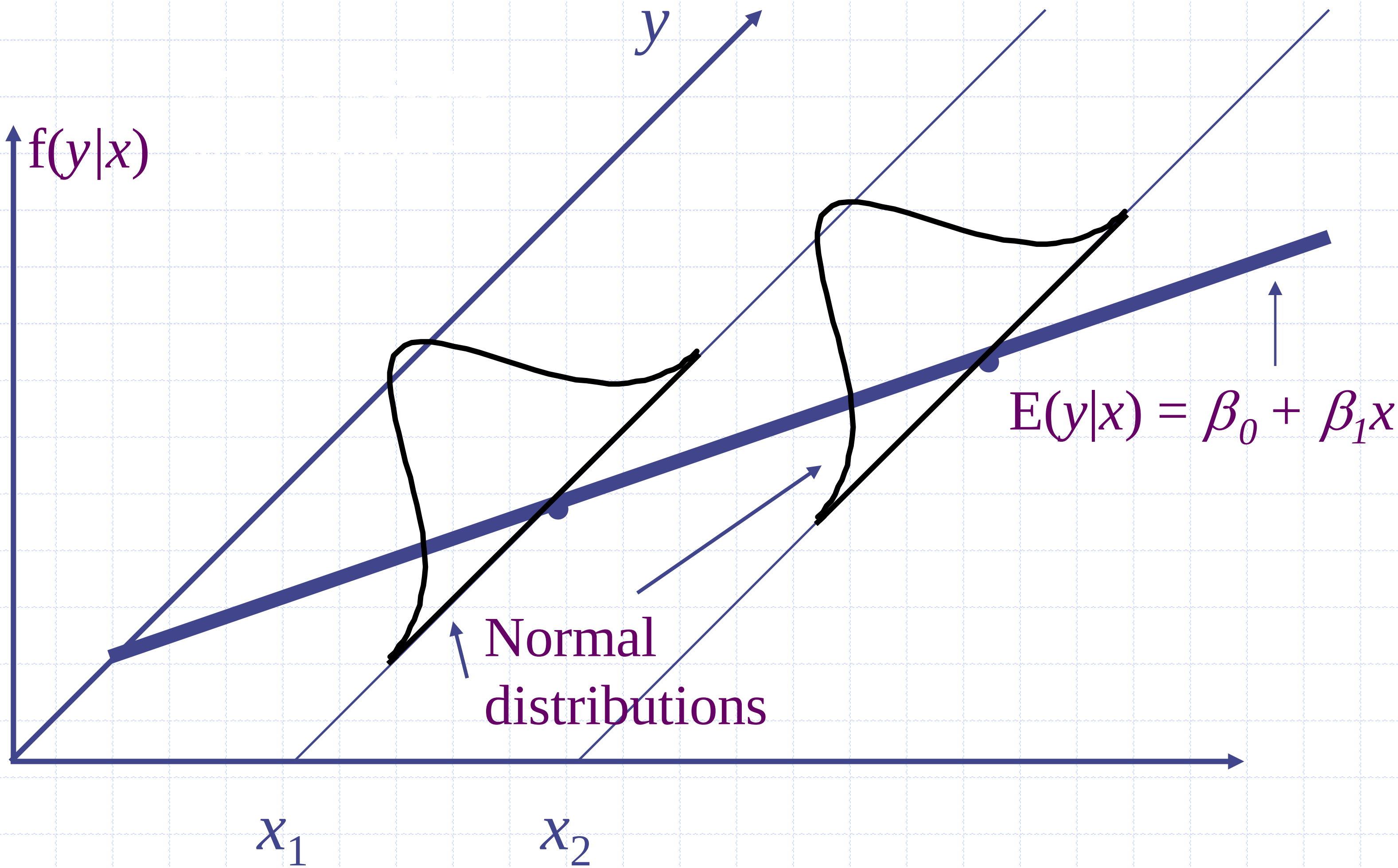
# The homoskedastic normal distribution with a single explanatory variable



# Normal Sampling Distributions

Under the CLM assumptions, conditional on the sample values of the independent variables

$\hat{\beta}_j \sim \text{Normal}\left[\beta_j, Var\left(\hat{\beta}_j\right)\right]$, so that

$$\left(\hat{\beta}_j - \beta_j\right) \Big/ sd\left(\hat{\beta}_j\right) \sim \text{Normal}(0,1)$$

$\hat{\beta}_{\mathrm{j}}$ is distributed normally because it is a linear combination of the errors

# The *t* Test

Under the CLM assumptions

$$\left(\hat{\beta}_j - \beta_j\right) \Big/ se\left(\hat{\beta}_j\right) \sim t_{n-k-1}$$

Note this is a *t* distribution (vs normal)

because we have to estimate $\sigma^2$ by $\hat{\sigma}^2$

Note the degrees of freedom : $n - k - 1$

# The *t* Test (cont)

- Knowing the sampling distribution for the standardized estimator allows us to carry out hypothesis tests
- Start with a null hypothesis
- For example, $H_0$: $\beta_j = 0$
- If accept null, then accept that $x_j$ has no effect on *y*, controlling for other *x*'s

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# The *t* Test (cont)

To perform our test we first need to form

"the" $t$ statistic for $\hat{\beta}_{\mathrm{j}} : t_{\hat{\beta}_j} \equiv \hat{\beta}_j / se\left(\hat{\beta}_j\right)$

We will then use our $t$ statistic along with a rejection rule to determine whether to accept the null hypothesis, $H_0$

# *t* Test: One-Sided Alternatives

- Besides our null, $H_0$, we need an alternative hypothesis, $H_1$, and a significance level
- $H_1$ may be one-sided, or two-sided
- $H_1: \beta_j > 0$ and $H_1: \beta_j < 0$ are one-sided
- $H_1: \beta_j \neq 0$ is a two-sided alternative
- If we want to have only a 5% probability of rejecting $H_0$ if it is really true, then we say our significance level is 5%

# One-Sided Alternatives (cont)

- Having picked a significance level, $\alpha$, we look up the $(1 - \alpha)^{th}$ percentile in a *t* distribution with $n - k - 1$ df and call this *c*, the critical value
- We can reject the null hypothesis if the *t* statistic is greater than the critical value
- If the *t* statistic is less than the critical value then we fail to reject the null

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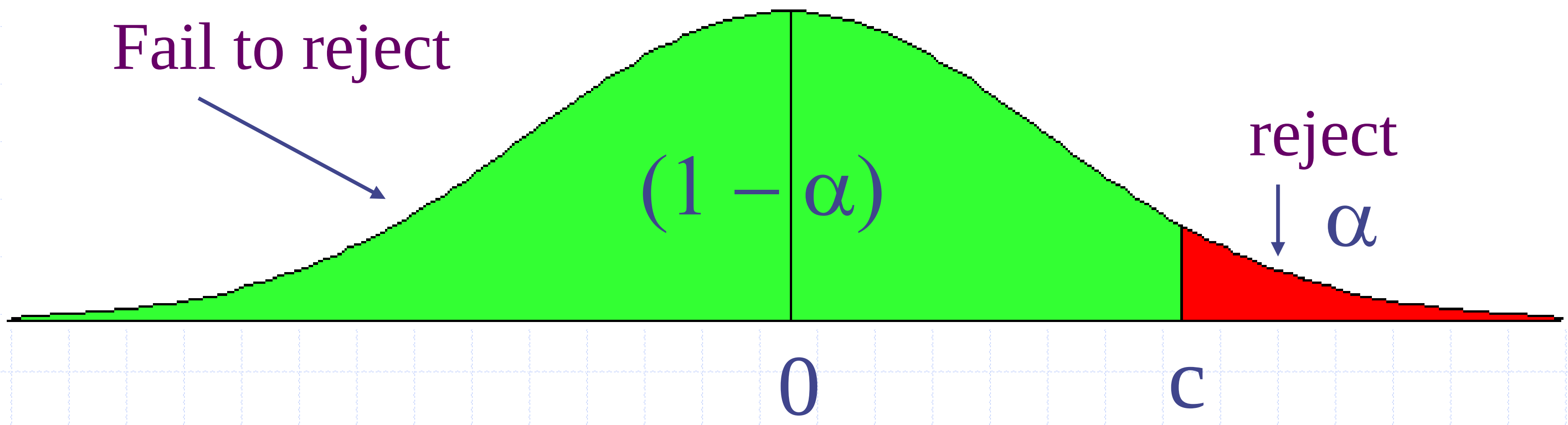
# One-Sided Alternatives (cont)

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_k x_{ik} + u_i$$

$H_0: \beta_j = 0$ $\qquad$ $H_1: \beta_j > 0$



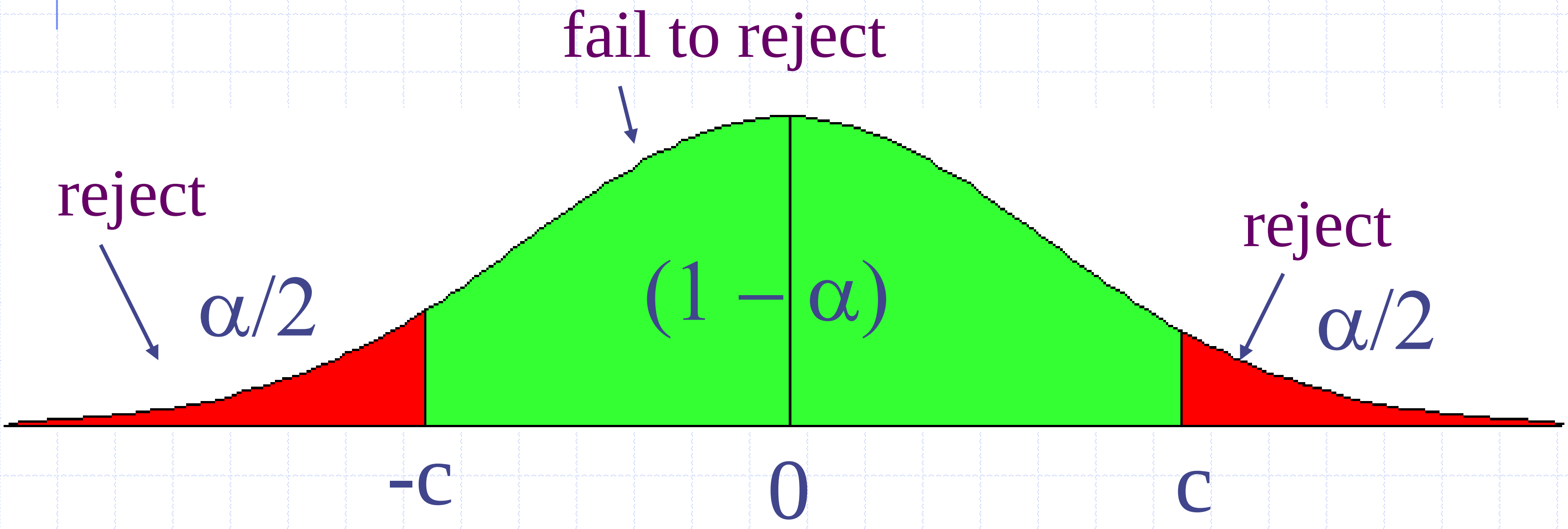
# One-sided vs Two-sided

- Because the *t* distribution is symmetric, testing $H_1$: $\beta_j < 0$ is straightforward. The critical value is just the negative of before
- We can reject the null if the *t* statistic < –*c*, and if the *t* statistic > than –*c* then we fail to reject the null
- For a two-sided test, we set the critical value based on $\alpha/2$ and reject $H_1$: $\beta_j \neq 0$ if the <u>absolute value</u> of the *t* statistic > *c*

# Two-Sided Alternatives

$$y_i = \beta_0 + \beta_1 X_{i1} + \ldots + \beta_k X_{ik} + u_i$$

$H_0: \beta_j = 0$ $\qquad$ $H_1: \beta_j \neq 0$



# Summary for $H_0$: $\beta_j = 0$

- Unless otherwise stated, the alternative is assumed to be two-sided
- If we reject the null, we typically say "$x_j$ is statistically significant at the $\alpha$ % level"
- If we fail to reject the null, we typically say "$x_j$ is statistically insignificant at the $\alpha$ % level"

# Testing other hypotheses

- A more general form of the *t* statistic recognizes that we may want to test something like $H_0$: $\beta_j = a_j$
- In this case, the appropriate *t* statistic is

$$t = \left(\hat{\beta}_j - a_j\right) \Big/ se\left(\hat{\beta}_j\right), \text{ where}$$

$$a_j = 0 \text{ for the standard test}$$

# Confidence Intervals

- Another way to use classical statistical testing is to construct a confidence interval using the same critical value as was used for a two-sided test
- A (1 - $\alpha$) % confidence interval is defined as

$$\hat{\beta}_j \pm c \bullet se\left(\hat{\beta}_j\right), \text{ where c is the } \left(1 - \frac{\alpha}{2}\right) \text{ percentile}$$

in a $t_{n-k-1}$ distribution

# Computing *p*-values for *t* tests

- An alternative to the classical approach is to ask, "what is the smallest significance level at which the null would be rejected?"
- So, compute the *t* statistic, and then look up what percentile it is in the appropriate *t* distribution – this is the *p*-value
- *p*-value is the probability we would observe the *t* statistic we did, if the null were true

# Stata and *p*-values, *t* tests, etc.

- Most computer packages will compute the *p*-value for you, assuming a two-sided test
- If you really want a one-sided alternative, just divide the two-sided *p*-value by 2
- Stata provides the *t* statistic, *p*-value, and 95% confidence interval for $H_0$: $\beta_j = 0$ for you, in columns labeled "t", "P > |t|" and "[95% Conf. Interval]", respectively

# Testing a Linear Combination

- Suppose instead of testing whether $\beta_1$ is equal to a constant, you want to test if it is equal to another parameter, that is $H_0 : \beta_1 = \beta_2$
- Use same basic procedure for forming a t statistic

$$t = \frac{\hat{\beta}_1 - \hat{\beta}_2}{se\left(\hat{\beta}_1 - \hat{\beta}_2\right)}$$

# Testing Linear Combo (cont)

Since

$$se\left(\hat{\beta}_1 - \hat{\beta}_2\right) = \sqrt{Var\left(\hat{\beta}_1 - \hat{\beta}_2\right)}, \text{ then}$$

$$Var\left(\hat{\beta}_1 - \hat{\beta}_2\right) = Var\left(\hat{\beta}_1\right) + Var\left(\hat{\beta}_2\right) - 2Cov\left(\hat{\beta}_1, \hat{\beta}_2\right)$$

$$se\left(\hat{\beta}_1 - \hat{\beta}_2\right) = \left\{\left[se\left(\hat{\beta}_1\right)\right]^2 + \left[se\left(\hat{\beta}_2\right)\right]^2 - 2s_{12}\right\}^{1/2}$$

where $s_{12}$ is an estimate of $Cov\left(\hat{\beta}_1, \hat{\beta}_2\right)$

# Testing a Linear Combo (cont)

- So, to use formula, need $s_{12}$, which standard output does not have
- Many packages will have an option to get it, or will just perform the test for you
- In Stata, after reg y x1 x2 … xk you would type test x1 = x2 to get a *p*-value for the test
- More generally, you can always restate the problem to get the test you want

# Example:

- Suppose you are interested in the effect of campaign expenditures on outcomes
- Model is $voteA = \beta_0 + \beta_1 \log(expendA) + \beta_2 \log(expendB) + \beta_3 prtystrA + u$
- $H_0$: $\beta_1 = -\beta_2$, or $H_0$: $\theta_1 = \beta_1 + \beta_2 = 0$
- $\beta_1 = \theta_1 - \beta_2$, so substitute in and rearrange $\Rightarrow voteA = \beta_0 + \theta_1 \log(expendA) + \beta_2 \log(expendB - expendA) + \beta_3 prtystrA + u$

# Example (cont):

- This is the same model as originally, but now you get a standard error for $\beta_1 - \beta_2 = \theta_1$ directly from the basic regression
- Any linear combination of parameters could be tested in a similar manner
- Other examples of hypotheses about a single linear combination of parameters:
  - $\beta_1 = 1 + \beta_2$ ; $\beta_1 = 5\beta_2$ ; $\beta_1 = -1/2\beta_2$ ; etc

# Multiple Linear Restrictions

- Everything we've done so far has involved testing a single linear restriction, (e.g. $\beta_1 = 0$ or $\beta_1 = \beta_2$ )
- However, we may want to jointly test multiple hypotheses about our parameters
- A typical example is testing "exclusion restrictions" – we want to know if a group of parameters are all equal to zero

# Testing Exclusion Restrictions

- Now the null hypothesis might be something like $H_0$: $\beta_{k-q+1} = 0, \dots, \beta_k = 0$
- The alternative is just $H_1$: $H_0$ is not true
- Can't just check each *t* statistic separately, because we want to know if the *q* parameters are jointly significant at a given level – it is possible for none to be individually significant at that level

# Exclusion Restrictions (cont)

- To do the test we need to estimate the "restricted model" without $x_{k-q+1}$, …, $x_k$ included, as well as the "unrestricted model" with all *x*'s included
- Intuitively, we want to know if the change in SSR is big enough to warrant inclusion of $x_{k-q+1}$, …, $x_k$

$$F \equiv \frac{\left(SSR_r - SSR_{ur}\right)/q}{SSR_{ur}/\left(n-k-1\right)}, \text{ where}$$

r is restricted and ur is unrestricted

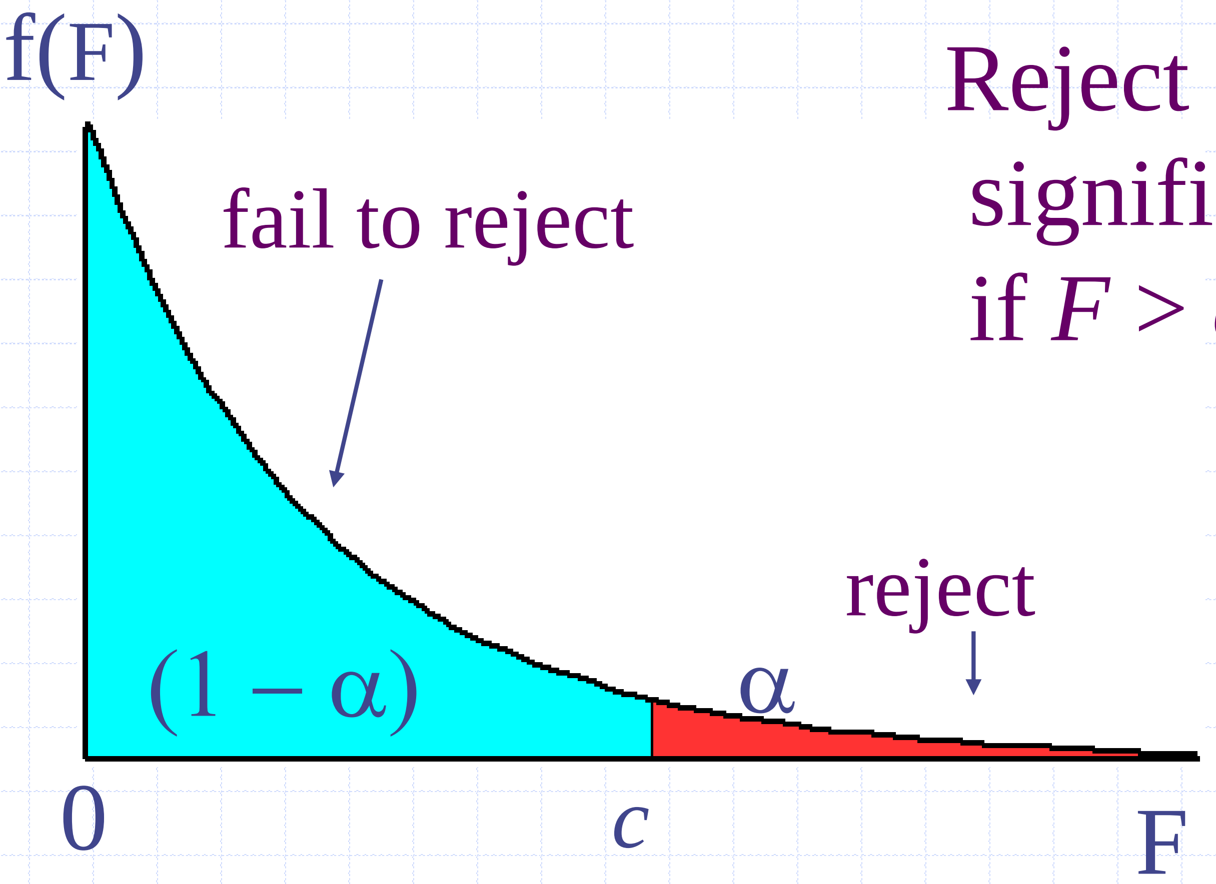
# The *F* statistic

- The *F* statistic is always positive, since the SSR from the restricted model can't be less than the SSR from the unrestricted
- Essentially the *F* statistic is measuring the relative increase in SSR when moving from the unrestricted to restricted model
- $q$ = number of restrictions, or $df_r - df_{ur}$
- $n - k - 1 = df_{ur}$

# The *F* statistic (cont)

- To decide if the increase in SSR when we move to a restricted model is "big enough" to reject the exclusions, we need to know about the sampling distribution of our *F* stat
- Not surprisingly, $F \sim F_{q,n-k-1}$, where $q$ is referred to as the numerator degrees of freedom and $n - k - 1$ as the denominator degrees of freedom

# The *F* statistic (cont)




Reject $H_0$ at α significance level if $F > c$

# The $R^2$ form of the $F$ statistic

- Because the SSR's may be large and unwieldy, an alternative form of the formula is useful
- We use the fact that SSR = SST(1 – $R^2$) for any regression, so can substitute in for $SSR_u$ and $SSR_{ur}$

$$F \equiv \frac{\left(R_{ur}^2 - R_r^2\right)/q}{\left(1 - R_{ur}^2\right)/(n-k-1)}, \text{ where again}$$

r is restricted and ur is unrestricted

# Overall Significance

- A special case of exclusion restrictions is to test $H_0: \beta_1 = \beta_2 = \ldots = \beta_k = 0$
- Since the $R^2$ from a model with only an intercept will be zero, the *F* statistic is simply

$$F = \frac{R^2 / k}{(1 - R^2) / (n - k - 1)}$$

# General Linear Restrictions

- The basic form of the *F* statistic will work for any set of linear restrictions
- First estimate the unrestricted model and then estimate the restricted model
- In each case, make note of the SSR
- Imposing the restrictions can be tricky – will likely have to redefine variables again

# Example:

- Use same voting model as before
- Model is $voteA = \beta_0 + \beta_1 \log(expendA) + \beta_2 \log(expendB) + \beta_3 prtystrA + u$
- now null is $H_0$: $\beta_1 = 1$, $\beta_3 = 0$
- Substituting in the restrictions: $voteA = \beta_0 + \log(expendA) + \beta_2 \log(expendB) + u$, so
- Use $voteA - \log(expendA) = \beta_0 + \beta_2 \log(expendB) + u$ as restricted model

# *F* Statistic Summary

- Just as with *t* statistics, p-values can be calculated by looking up the percentile in the appropriate *F* distribution
- Stata will do this by entering: display fprob($q$, $n - k - 1$, $F$), where the appropriate values of $F$, $q$,and $n - k - 1$ are used
- If only one exclusion is being tested, then $F = t^2$, and the *p*-values will be the same