The Solow Growth Model

Spring 2020

Solow growth model

Builds on the production model by adding a theory of capital accumulation

- Was developed in the mid-1950s by Robert Solow of MIT
- Was the basis for the Nobel Prize he received in 1987

Additions / differences with the model

- Capital stock is no longer exogenous
- Capital stock is now “endogenised”
- The accumulation of capital is a possible engine of long run growth

Setting up the model

Start with the production model

- Add an equation describing the accumulation of capital over time

The production function

- Cobb-Douglas
- Constant returns to scale in capital and labour
- Exponent of 1/3 on K

Variables are time subscripted (t): $$Y_t = F(K_t, L_t) = \bar{A}K_t^{1/3}L_t^{2/3}$$

Output can be used for consumption or investment: $$\underbrace{C_t}_{\text{Consumption}} + \underbrace{I_t}_{\text{Investment}} = \underbrace{Y_t}_{\text{Output}}$$

This is the resource constraint

- Assuming no imports, exports or government

Capital accumulation

Goods invested for the future determine the accumulation of capital

Capital accumulation equation:

$$\underbrace{K_{t+1}}_{\text{Next year's capital}} = \underbrace{K_t}_{\text{This year's capital}} + \underbrace{I_t}_{\text{Investment}} - \underbrace{\bar{d}}_{\text{Depreciation rate}} K_t$$

Depreciation rate

- The amount of capital that wears out each period
- Mathematically must be between 0 and 1 in this setting
- Often viewed as approximated 10 percent
- $\bar{d} = 0.10$

Change in capital stock

Change in capital stock defined as $\Delta K_{t+1} \equiv K_{t+1} - K_t$ thus $\Delta K_{t+1} = I_t - \bar{d}K_t$

The change in the stock of capital is investment less the capital that depreciates in production

To understand capital accumulation, we must assume the economy begins with a certain amount of capital K_0

- Suppose K_0=1,000
- and $\bar{d}$=0.10

TABLE 5.1

A Capital Accumulation Example

Time, t	Capital, K_t	Investment, I_t	Depreciation, $\bar{d}K_t$	Change in capital, ΔK_{t+1}
0	1,000	200	100	100
1	1,100	200	110	90
2	1,190	200	119	81
3	1,271	200	127	73
4	1,344	200	134	66
5	1,410	200	141	59

The last column is found by applying the capital accumulation equation: $\Delta K_{t+1} = I_t - \bar{d}K_t$. That is, it is computed by taking the difference between the two prior columns. The next period's capital stock is then the sum of K_t and ΔK_{t+1}.

Investment and Labour

Agents consumer a fraction of output and invest the rest $I_t = \bar{s}Y_t$

Fraction invested

Therefore consumption is the share of output not invested $C_t = (1 - \bar{s})Y_t$

To keep things simple, labour demand and supply not included

The amount of labour in the economy is given exogenously at a constant level $L_t = \bar{L}$

Summary of the Solow model

Unknowns / endogenous variables: Y_t, K_t, L_t, C_t, I_t, parameters $\bar{A}, \bar{s}, \bar{d}, \bar{L}, \overline{K_0}$

Relationship	Equation
Production function	$Y_t = \bar{A}K_t^{1/3}L_t^{2/3}$
Capital accumulation	$\Delta K_{t+1} = I_t - \bar{d}K_t$
Labour supply	$L_t = \bar{L}$
Resource constraint	$C_t + I_t = Y_t$
Allocation of resources	$I_t = \bar{s}Y_t$

Differences between Solow and production models:

- Dynamics of capital accumulation added
- Left out capital and labour markets, along with their prices

Framework of the model

- Variables
  - Capital K_t
  - Labor L_t
  - Output Y_t
  - Consumption C_t
  - Investment I_t
- Parameters
  - Depreciation rate δ (0.1)
  - Savings rate s (0.2-0.5)
  - Size of labor force $\bar{L}$
  - Total factor productivity A
  - Initial amount of capital K_0

5 equations, 5 variables

- Equations
  - Technology:
    $$Y_t = AK_t^{\alpha}L_t^{1-\alpha}$$
  - Resource constraint:
    $$Y_t = C_t + I_t$$
  - Capital accumulation:
    $$K_{t+1} = K_t + I_t - \delta K_t$$
  - Saving/investment function:
    $$I_t = sY_t$$
  - Fixed labor force:
    $$L_t = \bar{L}$$

Putting labour and capital markets back in

If we added equations for the wage and rental rate of capital:

- The MPL and MPK would pin them down
- Omitting them changes nothing

Depreciation now gets subtracted

$$\frac{dF(K,L)}{dL} \equiv MPL = w$$

$$\frac{dF(K,L)}{dK} - \bar{d} \equiv MPK = r$$

The real interest rate

- The amount a person can earn by saving one unit of output for a year
- Or, the amount a person must pay to borrow one unit of output for a year
- Measured in constant £, not nominal £

A unit of investment becomes a unit of capital

- The return on saving must equal the rental price of capital
- The real interest rate equals the rental rate of capital, which equals the MPK

Solving the Solow model

The model needs to be solved at every point in time, which cannot be done algebraically

Two ways to make progress

- Show a graphical solution
- Solve the model in the long run

We start by combining equations to go as far as we can with algebra

Combine the investment allocation and capital allocation equation $$\Delta K_{t+1} = \bar{s}Y_t - \bar{d}K_t$$

Net investment = Gross investment – Depreciation

Substitute the fixed amount of labour into the production function $$Y_t = \bar{A}K_t^{1/3}\bar{L}^{2/3}$$

We have reduced the system into two equations and two unknowns Y_t, K_t

The Solow diagram

Gross investment

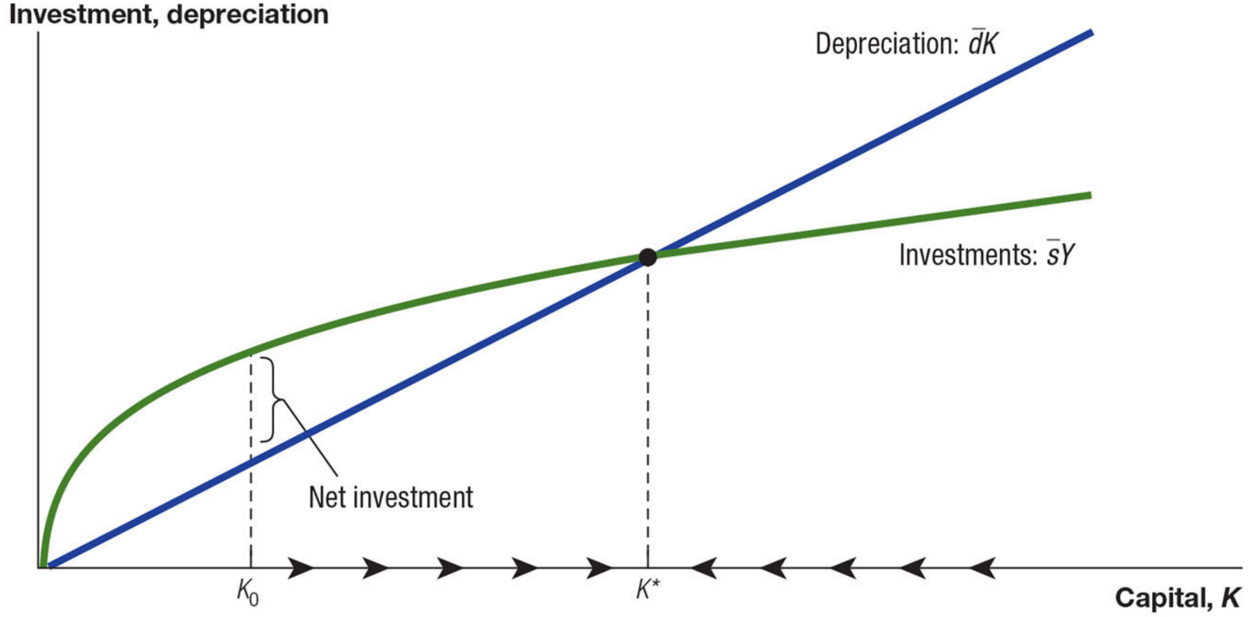
Depreciation

Plots the two terms that govern the change in the capital stock $\bar{s}Y_t$ $\bar{d}K_t$

Gross investment is the production function scaled by the investment rate $\bar{s}Y_t = \bar{s}\bar{A}K_t^{1/3}\bar{L}^{2/3}$

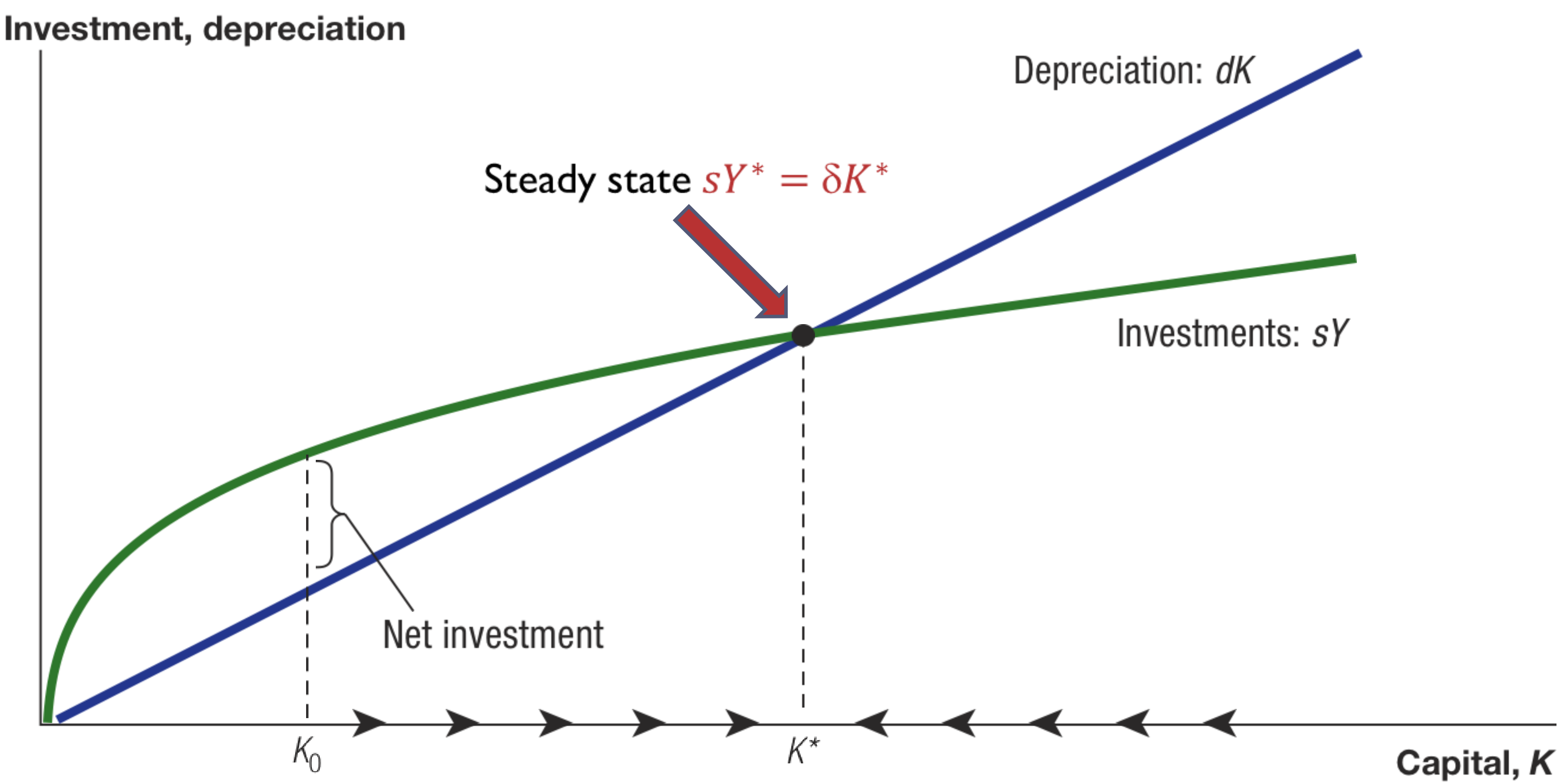
FIGURE 5.1

The Solow Diagram



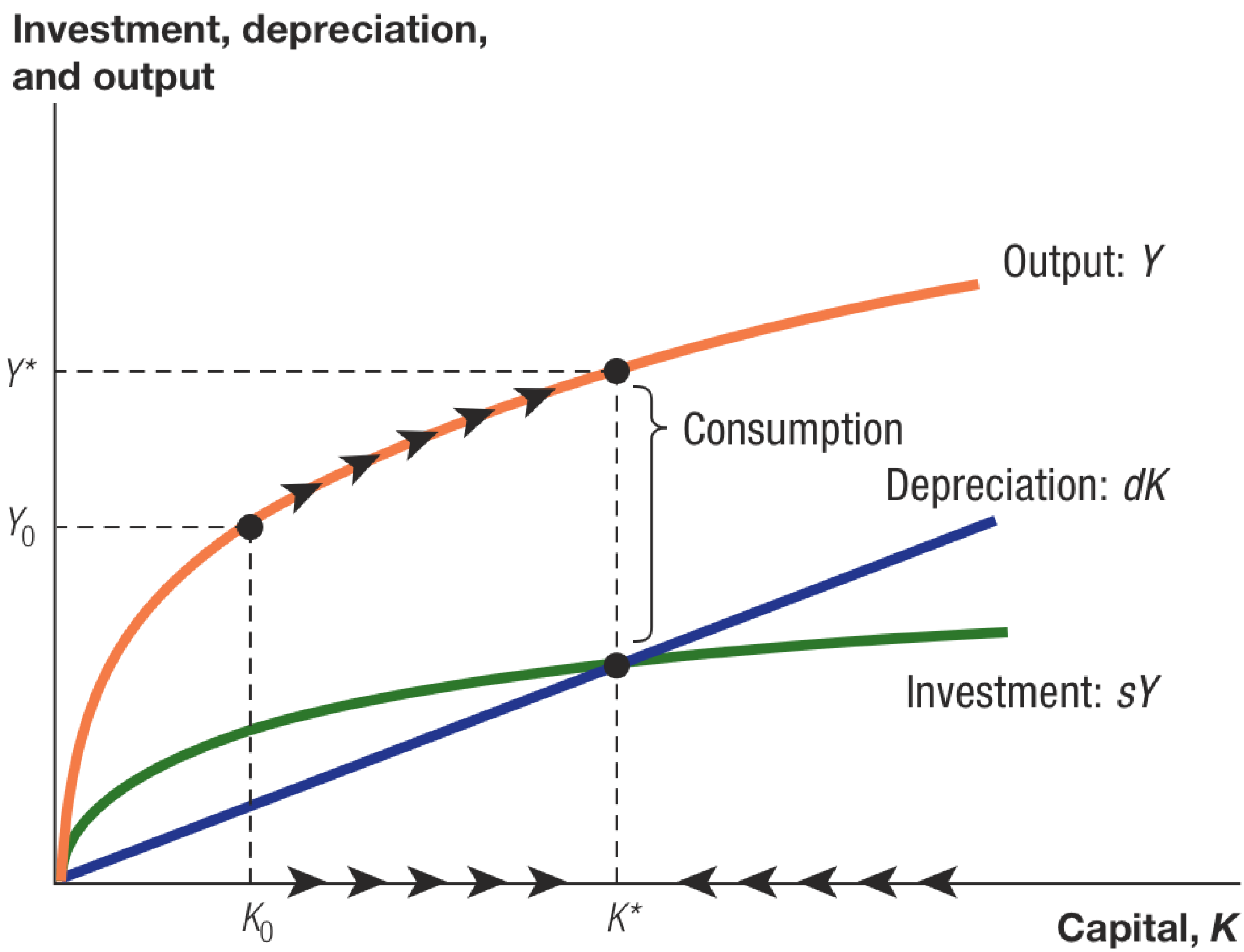
Solow diagram

$$\Delta K_{t+1} = sY_t - \delta K_t \qquad Y_t = AK_t^{\alpha} L^{1-\alpha}$$



Source: Jones – Macroeconomics

Solow diagram – adding consumption



Source: Jones – Macroeconomics

Using the Solow diagram

If the amount of investment is greater than the amount of depreciation the capital stock will increase until investment equals depreciation

- here, the change in capital is equal to 0
- the capital stock will stay at this value of capital forever
- this is called the steady state

If depreciation is greater than investment, the economy converges to the same steady state as above

Dynamics of the model

- When not in steady state, the economy exhibits a movement of capital towards the steady state
- At the rest point of the economy, all endogenous variables are steady
- Transition dynamics take the economy from its initial level of capital to the steady state

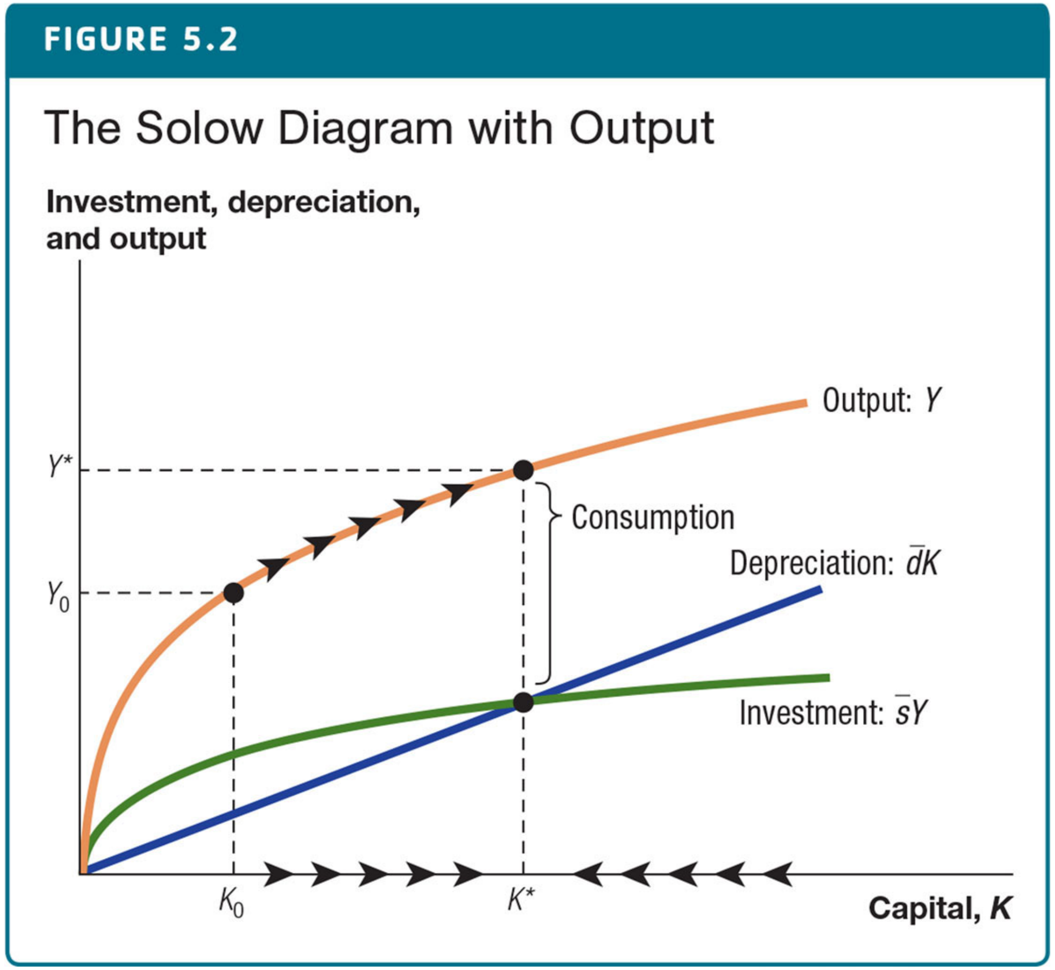
Output and consumption in the Solow diagram

As K moves to its steady state by transition dynamics, output will also move to its steady state

$$Y_t = \bar{A}K_t^{1/3}\bar{L}^{2/3}$$

Consumption can also be seen in the diagram since it is the difference between output and investment

$$C_t = Y_t - I_t$$



Solving mathematically for steady-state capital

In the steady state, investment equals depreciation $\bar{s}Y^* = \bar{d}K^*$

Substitute into the production function $\bar{s}\bar{A}{K^*}^{1/3}\bar{L}^{2/3} = \bar{d}K^*$

Solve for $K^* = \left(\frac{\bar{s}\bar{A}}{\bar{d}}\right)^{3/2} \bar{L}$

The steady-state level of capital is

- Positively correlated with the investment rate, the size of the workforce and the productivity of the economy
- Negatively correlated with the depreciation rate

Solving mathematically for steady-state output

Plug K^* into the production function to get $Y^* = \left(\frac{\bar{s}}{\bar{d}}\right)^{1/2} \bar{A}^{3/2}\bar{L}$

- Higher steady-state production caused by higher productivity and investment rate
- Lower steady-state production caused by faster depreciation

Divide both sides by labour to get output per person (*y*) in steady state

$$y^* \equiv \frac{Y^*}{L^*} = \left(\frac{\bar{s}}{\bar{d}}\right)^{1/2} \bar{A}^{3/2}$$

Note the exponent on productivity is different here (3/2) than in the production model (1)

- Higher productivity has additional effects in the Solow model by leading the economy to accumulate more capital

Understanding the steady state

The economy reaches a steady state because investment has diminishing returns

- The rate at which production and investment rise is smaller as the capital stock increases

Also, a constant fraction of the capital stock depreciates each period

- Depreciation is not diminishing as capital increases

Eventually, net investment is zero

- The economy rests in steady state

There is no long-run economic growth in the Solow model

- In the steady state, growth stops
- Output, capital, output per person and consumption per person are all constant
- Capital accumulation cannot be the engine of long-run economic growth
- Saving and investment are beneficial in the short run but do not sustain long-run growth, due to diminishing returns

Experiments in the Solow model – an increase in $\bar{s}$

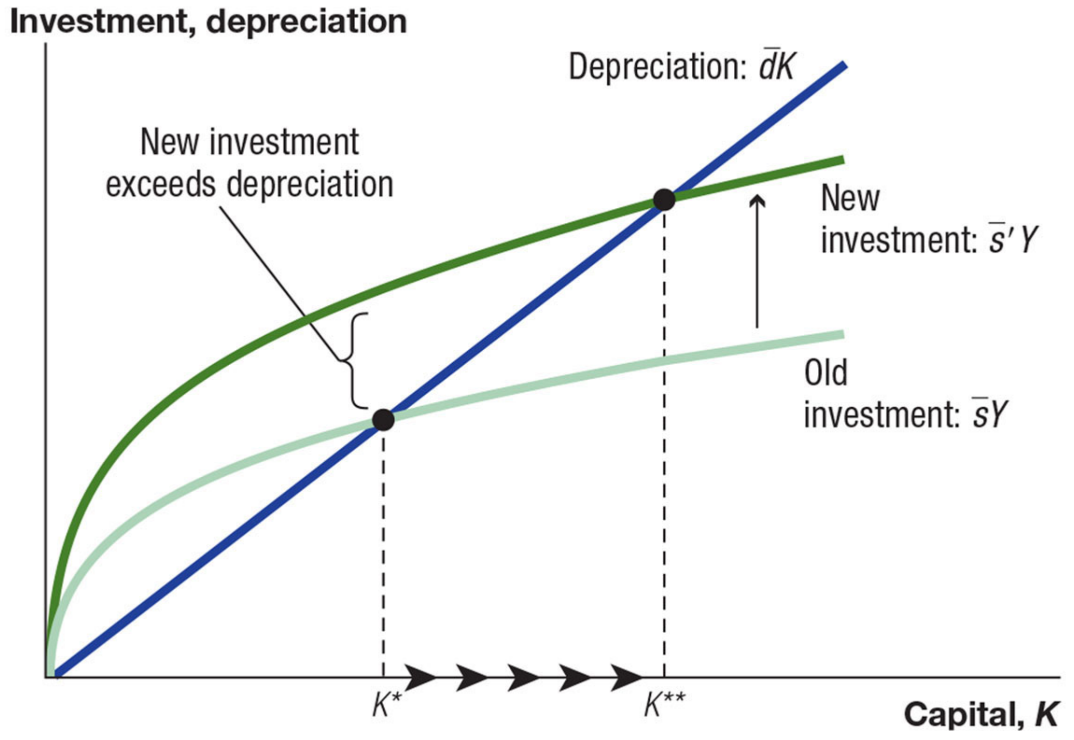
$$\bar{s} \to \bar{s}'$$

Suppose the investment rate increases permanently for exogenous reasons

- The investment curve $\bar{s}Y \to \bar{s}'Y$ rotates upwards
- The depreciations curve $\bar{d}K$ remains unchanged
- The capital stock increases by transition dynamics to reach the new steady state because investment exceeds depreciation
- The new steady state $\bar{s}'Y = \bar{d}K$ is located to the right

FIGURE 5.4

An Increase in the Investment Rate



What happens to output in response to the increase in the investment rate?

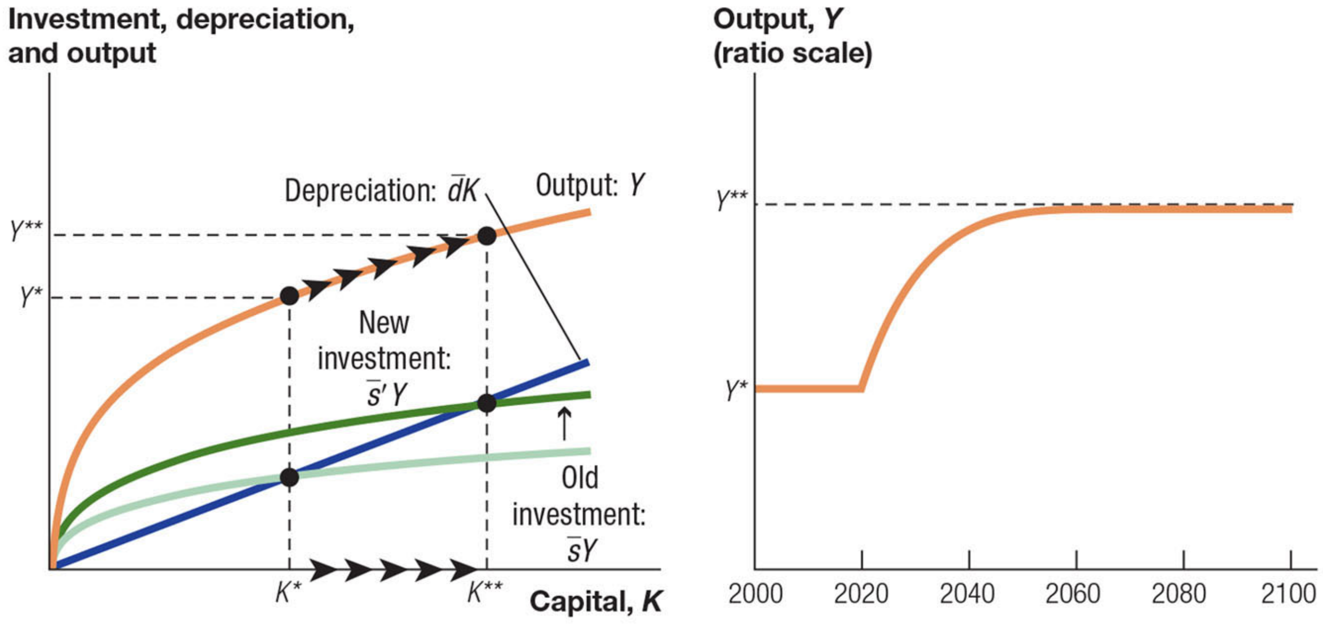
The rise in investment leads capital to accumulate over time

This higher capital causes output to rise as well

Output increases from its initial steady-state level Y^* to the new steady state Y^{**}

FIGURE 5.5

The Behavior of Output after an Increase in $\bar{s}$



Experiments in the Solow model – an increase in $\bar{d}$

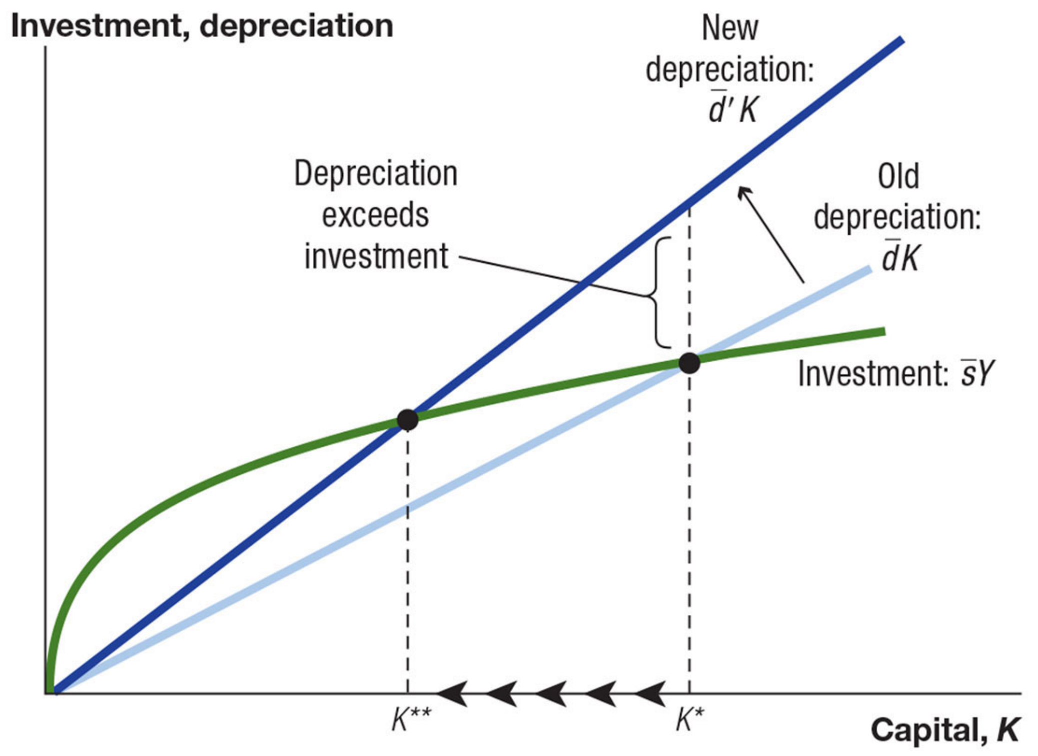
$$\bar{d} \rightarrow \bar{d}'$$

Suppose the depreciation rate is exogenously shocked to a higher rate

- The depreciations curve $\bar{d}K$ rotates upwards
- The investment curve $\bar{s}Y \rightarrow \bar{s}'Y$ remains unchanged
- The capital stock declines by transition dynamics to reach the new steady state because depreciation exceeds investment
- The new steady state $\bar{s}'Y = \bar{d}K$ is located to the left

FIGURE 5.6

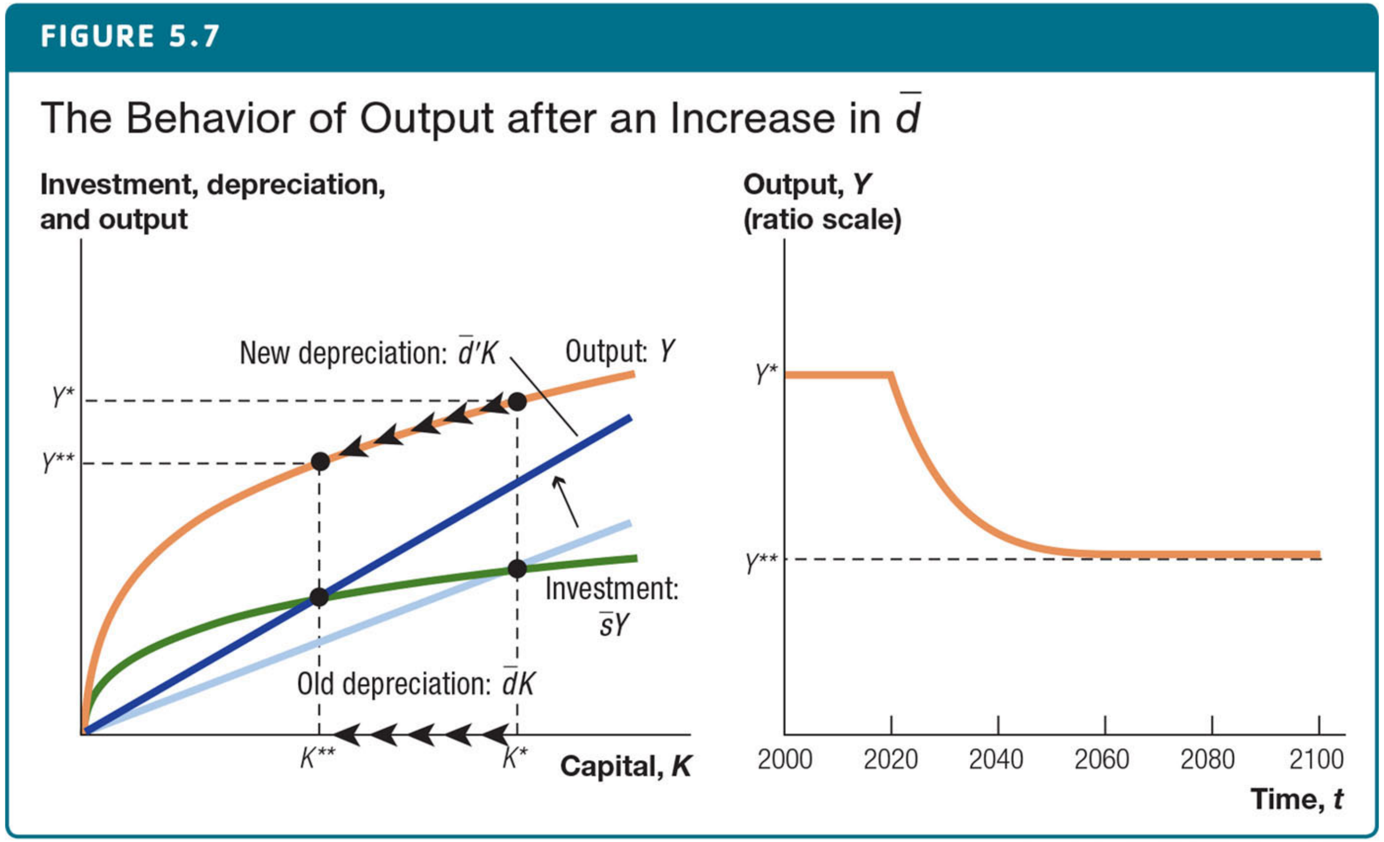
A Rise in the Depreciation Rate



What happens to output in response to the increase in the depreciation rate?

The decline in capital reduces output

Output declines rapidly at first, and then gradually settles down to its new, lower steady state level Y^{**}



Strengths of the Solow Model

1. Theory of country's income in the long run (in steady state)
   - High rate of investment, high TFP
   - investment could include investment in human capital
     - e.g., education, on-the-job training

2. Transition dynamics:
   - Helps us understand differences in growth rates
   - Country further below its steady state grows faster
   - If determinants of long-run income increase
     - e.g., investment rates, TFP

     $\Rightarrow$ higher short run growth
   - Catchup growth: China, Ireland, South Korea

Weaknesses of the Solow Model

1. The main mechanism is investment in physical capital
    - Differences in investment rate correlated with GDP/capita
    - But this only explains small fraction of income differences
    - TFP differences are much more important
        - These remain a bit of a mystery

Weaknesses of the Solow Model

2. Why do countries have different TFP and investment rates?

- South Korea vs. Philippines
  - In 1960s, each had $\approx$ 10% of US income
  - Since, South Korea grew at 6% per year, Philippines 1.6% per year
  - In South Korea, investment rate and productivity increased
    - investment rate $\approx$ 40%, Grew to 50-75% of US income
  - In Philippines it did not
    - Remained at roughly 10% of US income

- Solow model does not explain why investment rate increased
  - Literature has extended Solow model
    - savings rate depends on patience, investment taxes/subsidies
  - But ultimately, without explaining TFP differences, can only explain part of the income differences

Weaknesses of the Solow Model

3. Solow model does not provide theory of long run growth
    - Mechanism: saving leads to investment in factories, machines
    - but diminishing returns to capital
        - $\Rightarrow$ capital accumulation itself cannot sustain growth

- Growth is exogenous: does not explain why growth happens