

---

# **Principles of Finance**

## **How to Calculate Present Value**

# Overview

---

## ● How To Calculate Present Values

- Valuing Long-Lived Assets
  - Absence of Arbitrage
  - PV Shortcuts – Perpetuities and Annuities
  - Internal Rate of Return
  - Compound Interest & Present Values
  - Nominal and Real Interest Rates
- Based on Chapters 2, 3.5, 5 and 6.1 in BMA
  - See also Chapters 5.1 and 5.2 in BKM

# Return Definitions

---

- *Discount rate*: Rate used to calculate the present value of future cash flows
- *(Opportunity) cost of capital*: Expected return that is foregone by investing in a project rather than in comparable investment opportunity, e.g., financial securities
- *Required rate of return (hurdle rate)*: Rate required from an investment to undertake it
- Required return, hurdle rate and opportunity cost of capital are different names for the same concept

# Discount Factor

---

- Define Discount Factor (DF) as PV of £1:

$$DF = \frac{1}{(1 + r_t)^t}$$

- Discount Factors can be used to compute the present value of any cash flow realizing at time t:

$$PV = DF \times C_t = \frac{C_t}{(1 + r_t)^t}$$

# Present Values

---

- Since PVs express value in current £, they can be added up to evaluate cash flows accruing at different times

$$PV = \frac{C_1}{1 + r_1} + \frac{C_2}{(1 + r_2)^2} + \dots + \frac{C_T}{(1 + r_T)^T}$$

- A shorthand of the Discounted Cash Flow (DCF) formula is

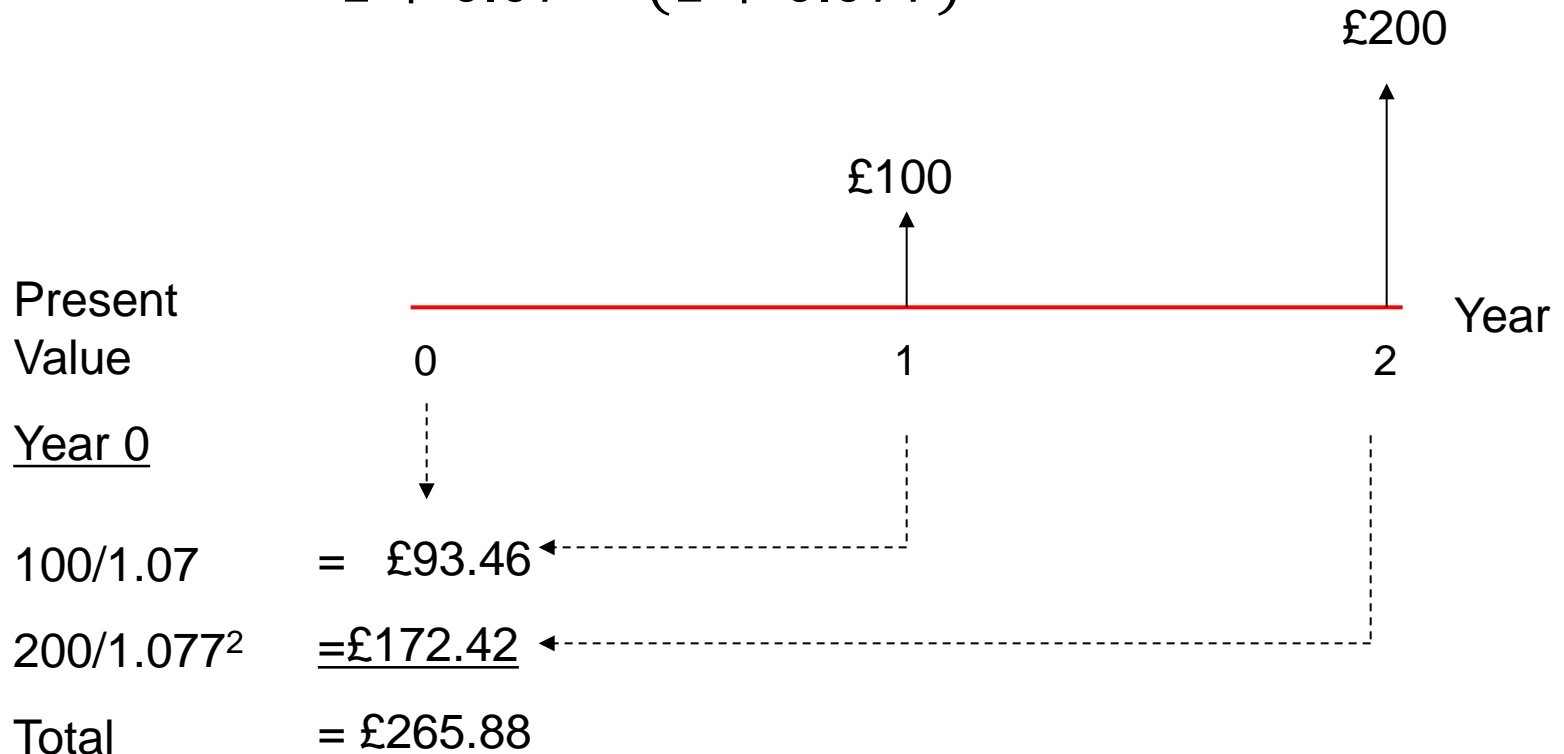
$$PV = \sum_{t=1}^T \frac{C_t}{(1 + r_t)^t}$$

- In general,  $r_1, r_2, r_3, \dots$  can be different

# Present Values

---

$$PV = \frac{100}{1 + 0.07} + \frac{200}{(1 + 0.077)^2} = 265.88$$



# Example

---

- Back to land/office development example.
  - Construction takes now two years, requiring an initial payment of £120,000 plus £50,000 for the land, a payment of £100,000 in a year from now, and a final payment of £100,000 at the end of the second year
- Despite the delay the building can be sold for £420,000
- What are cash flows, PVs and NPV?

Year 0	Year 1	Year 2
-170,000	-100,000	+320,000

## Example (cont'd)

---

- The PV of one dollar received in year  $t$  is the Discount Factor (DF) for year  $t$ .
- Given  $r = 5\%$  the discount factors for year 1 and year 2 are:

$$DF_1 = \frac{1}{1 + 0.05} = 0.9524$$

$$DF_2 = \frac{1}{(1 + 0.05)^2} = 0.9070$$



## Example (cont'd)

---

Period	Discount Factor	Cash Flow	Present Value
0	1.00	-170,000	-170,000
1	0.95	-100,000	-95,238
2	0.91	+320,000	+290,249
		NPV=	+25,011

---

# Absence of Arbitrage

---

- While interest rates  $r_t$  may differ across periods, discount factors must be smaller for cash flow streams accruing further in the future
  - A £ tomorrow is worth more than a £ the day after tomorrow
- Suppose to the contrary and consider time-1 and time-2 discount factors:
$$DF_1 = 1/(1.2)=0.83 \quad \text{and} \quad DF_2 = 1/(1.07)^2 = 0.87$$
- Hence, £1 in year 2 would be more valuable than £1 in year 1

# Absence of Arbitrage

---

- Such discount factors imply arbitrage opportunities, that is, allow to make (unlimited) profits for sure
- Arbitrage strategy
  - $t=0$ : borrow  $1048 = 1200/(1+0.07)^2$  for 2 years, lend 1000 for 1 year at 20%
  - $t=1$ : Store repaid £1200 in cash until year 2
  - $t=2$ : Repay loan £1200
  - Net (overall) gain (from  $t=0$ ): 48

# Absence of Arbitrage

---

- Arbitrage opportunities (would) offer sure way of making money. In well-functioning financial markets such opportunities are quickly/immediately eliminated
- Interest rates (or prices) adjust to eliminate arbitrage. Thus, absence of arbitrage imposes restrictions on the discount factors
- Important implication of no-arbitrage is law of one price: Securities that have identical cash flows should be traded at the same price in well-functioning markets

## Quick Question

---

- A project generates cash flows of £432,000 in year 1, £437,000 in year 2, and £330,000 in year 3. The cost of capital is 15%. What is the project's PV?

# Answer

---

# Shortcuts

---

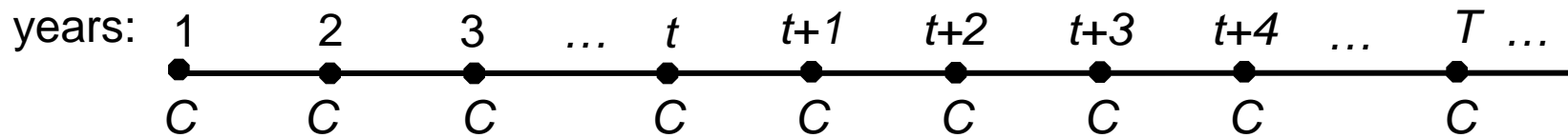
- When cash flows accrue over many, many periods, calculating PVs can get very tiresome
- There are – sometimes - shortcuts that make it easy to figure out the PVs of assets paying off in many periods
- The tools allowing us to perform calculations quickly are:
  - Perpetuity
  - Annuity
  - Perpetuity with growth
  - Annuity with growth

# Perpetuity

---

- Perpetuity: Asset/security that pays a constant cash flow each period to perpetuity:

Perpetuity with first payment in year 1



- Present value at time 0 is given by

$$PV = \frac{C}{r}$$



# Perpetuity

---

Derivation of the formula using infinite series

$$PV = a(1 + x + x^2 + x^3 + x^4 + \dots + x^t + \dots), \text{ where } x = \frac{1}{1+r} \text{ and } a = \frac{C}{1+r}$$

$$PV \cdot x = a \cdot (x + x^2 + x^3 + x^4 + \dots + x^t + \dots)$$

$$PV - PV \cdot x = a$$

$$PV(1 - x) = a$$

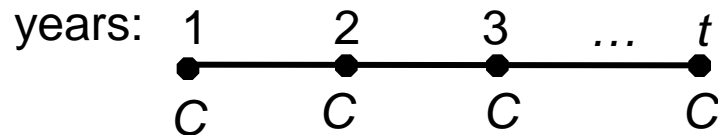
$$PV = \frac{C}{1+r} \cdot \frac{1}{1 - \frac{1}{1+r}} = \frac{C}{r}$$

# Annuity

---

- Annuity: Asset/security that pays a fixed cash flow each period for a specified number of years:

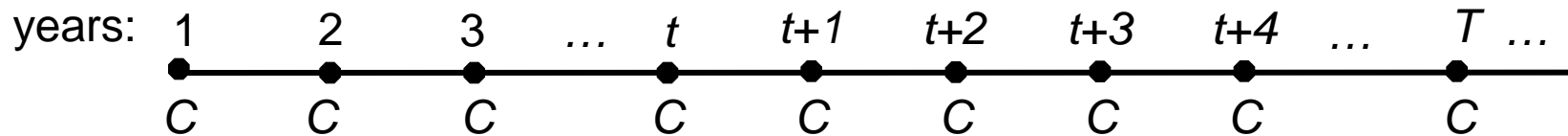
Annuity from year 1 to year  $t$



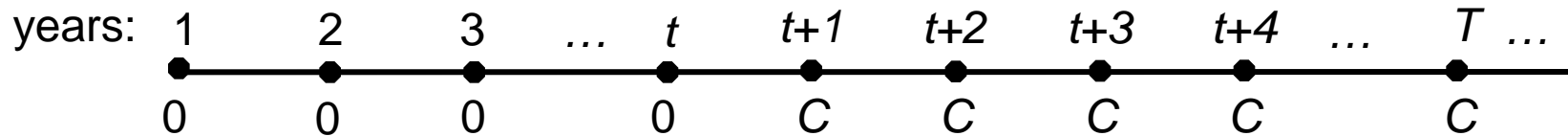
- The PV of an annuity is equal to the value difference of a perpetuity with its first payment in year 1 and a perpetuity with its first payment in year  $t+1$

# Annuity

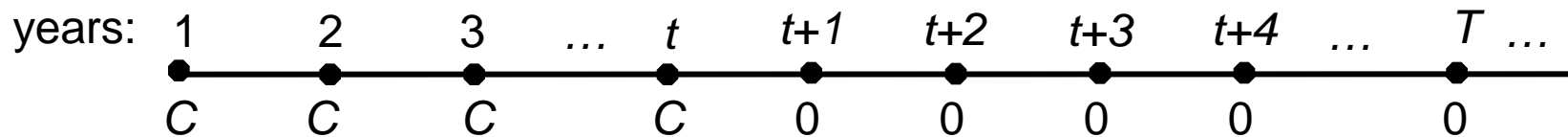
Perpetuity with first payment at year 1 ( $P_1$ )



Perpetuity with first payment at year t+1 ( $P_{t+1}$ )



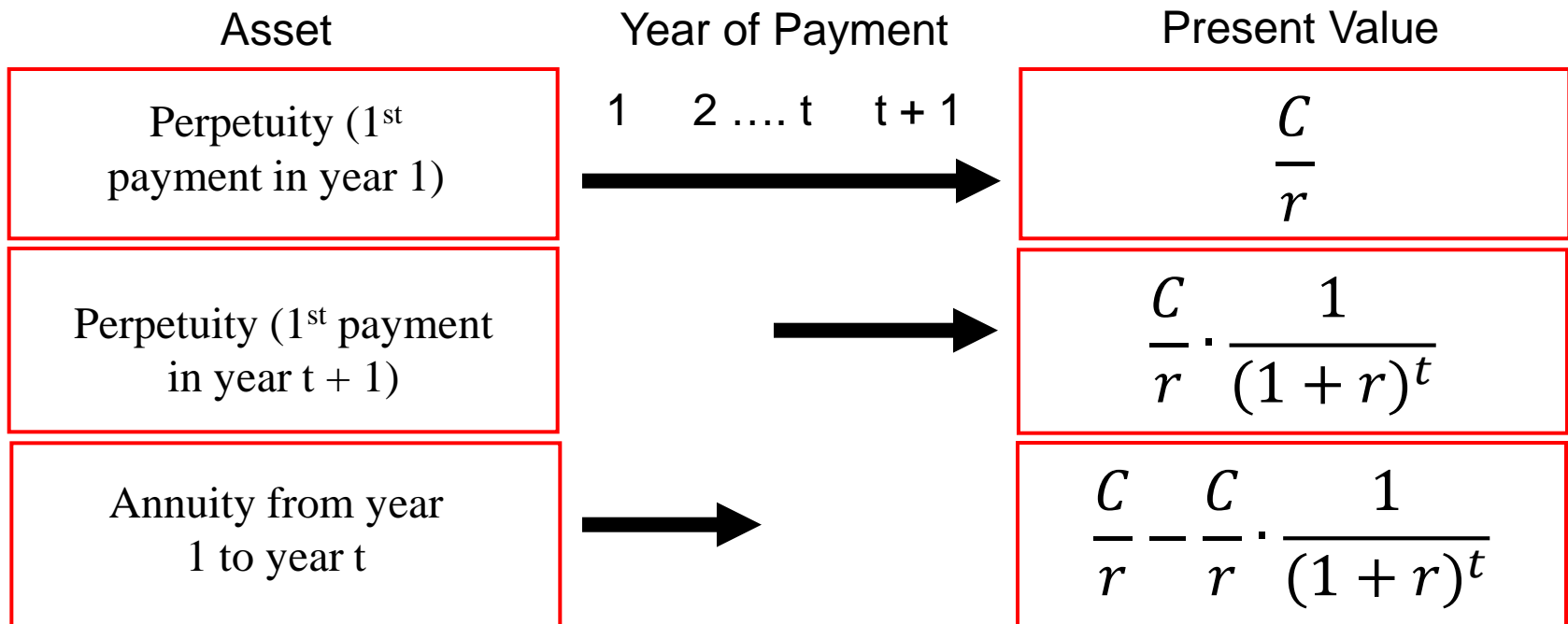
Annuity from year 1 to year t (A)



$$A = P_1 - P_{t+1}$$

# Annuity

$$PV(\text{annuity}) = PV(\text{perpetuity}) - \frac{1}{(1+r)^t} PV(\text{perpetuity})$$



$$PV \text{ of annuity} = C \times \left[ \frac{1}{r} - \frac{1}{r(1+r)^t} \right]$$

# Example

---

- Leasing a car requires monthly payments of £300 during four years, payable at the end of each month. If the opportunity cost of capital is 0.5% per month, what is the PV of the cost of the lease?

$$\text{Lease Cost} = 300 \times \left[ \frac{1}{.005} - \frac{1}{.005(1 + .005)^{48}} \right] = \text{£}12,774.10$$

where  $\left[ \frac{1}{.005} - \frac{1}{.005(1 + .005)^{48}} \right]$  is the (48 month) annuity factor

# Quick Question

---

- In May 1998 a retired couple bought a lottery ticket and won a record £194 million. However, this sum was to be paid in 25 equal instalments. If the first instalment is paid out immediately (annuity due), and the interest rate was 9%, what is the PV of the prize?

# Answer

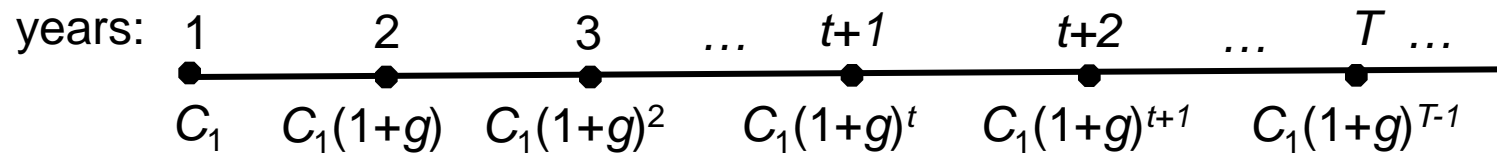
---

# Perpetuity With Growth

---

- Consider a security which pays  $C_1$  next period, and then the payment grows at rate  $g$  forever

Perpetuity with growth with first payment in year 1



- Given  $r$  is fixed and  $g < r$ , the present value at time 0 is given by

$$PV = \frac{C_1}{r - g}$$



# Annuity With Growth

---

- Consider a security which pays  $C_1$  next period, and then the payments grow at a rate  $g$  until period  $t$  after which payments terminate
- Its PV is equivalent to the PV of a perpetuity with growth minus the PV of that perpetuity with growth starting at time  $t$ .

$$PV = \frac{C_1}{r - g} - \left( \frac{1 + g}{1 + r} \right)^t \frac{C_1}{r - g}$$

# Internal Rate of Return

---

- The *internal rate of return* (IRR) is defined as the discount/return rate that makes NPV=0:

$$\text{NPV} = C_0 + \frac{C_1}{1 + \text{IRR}} + \frac{C_2}{(1 + \text{IRR})^2} + \dots + \frac{C_T}{(1 + \text{IRR})^T} = 0$$

- With a single payoff, IRR is also the rate of return

$$\text{NPV} = -C_0 + \frac{C_1}{1 + \text{IRR}} = 0$$

and hence  $\text{IRR} = \frac{C_1 - C_0}{C_0}$

# Internal Rate of Return

---

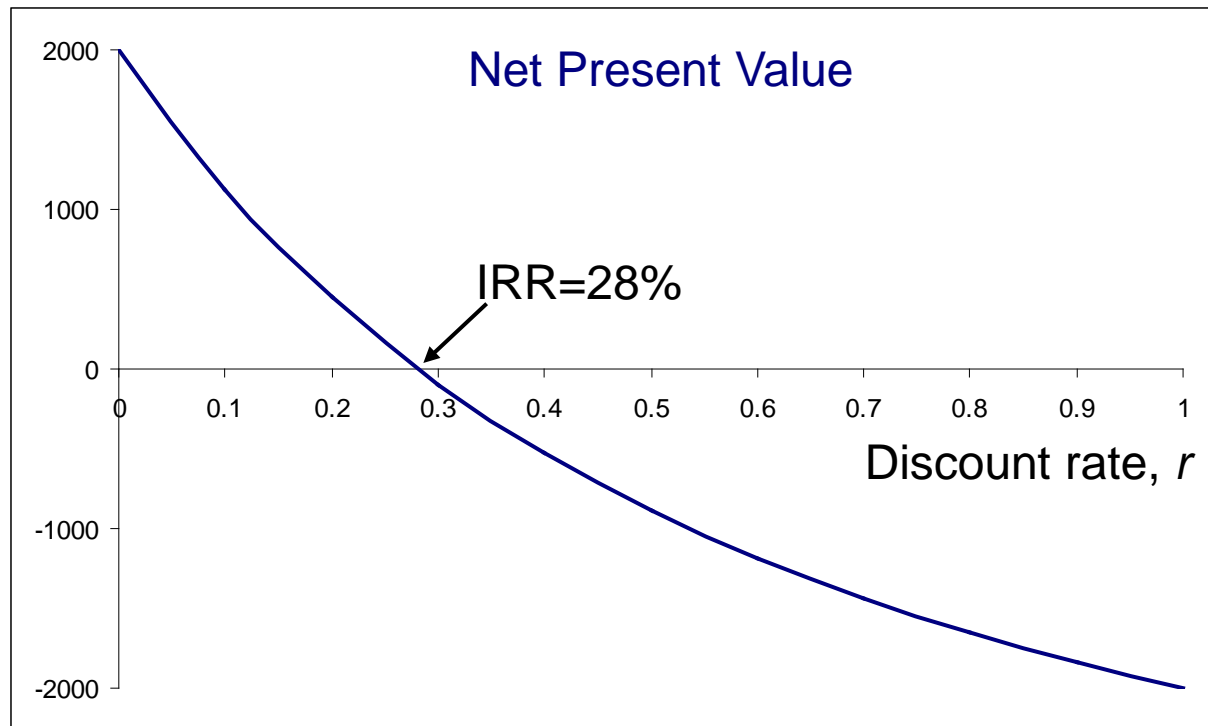
- Consider a project that has the following cash flows:

Cash Flows		
$C_0$	$C_1$	$C_2$
-4,000	+2,000	+4,000

- IRR can be found by trial and error or by using numerical methods/financial calculator

$$NPV = -4,000 + \frac{2,000}{1 + IRR} + \frac{4,000}{(1 + IRR)^2} = 0$$

# Internal Rate of Return



- IRR rule: Accept projects with  $IRR > r$ , because  $NPV > 0$

# Pitfalls of Using IRR

---

- The IRR method cannot distinguish between investing (or lending) and borrowing

Project	$C_0$	$C_1$	IRR	NPV@10%
A	-1,000	+1,500	+50%	+364
B	+1,000	-1,500	+50%	-364

- When borrowing IRR should be less not more than opportunity cost of capital.
  - When borrowing, NPV is increasing – rather than decreasing - with the discount rate
- NPV rule provides correct answer for both lending and borrowing

# Pitfalls of Using IRR

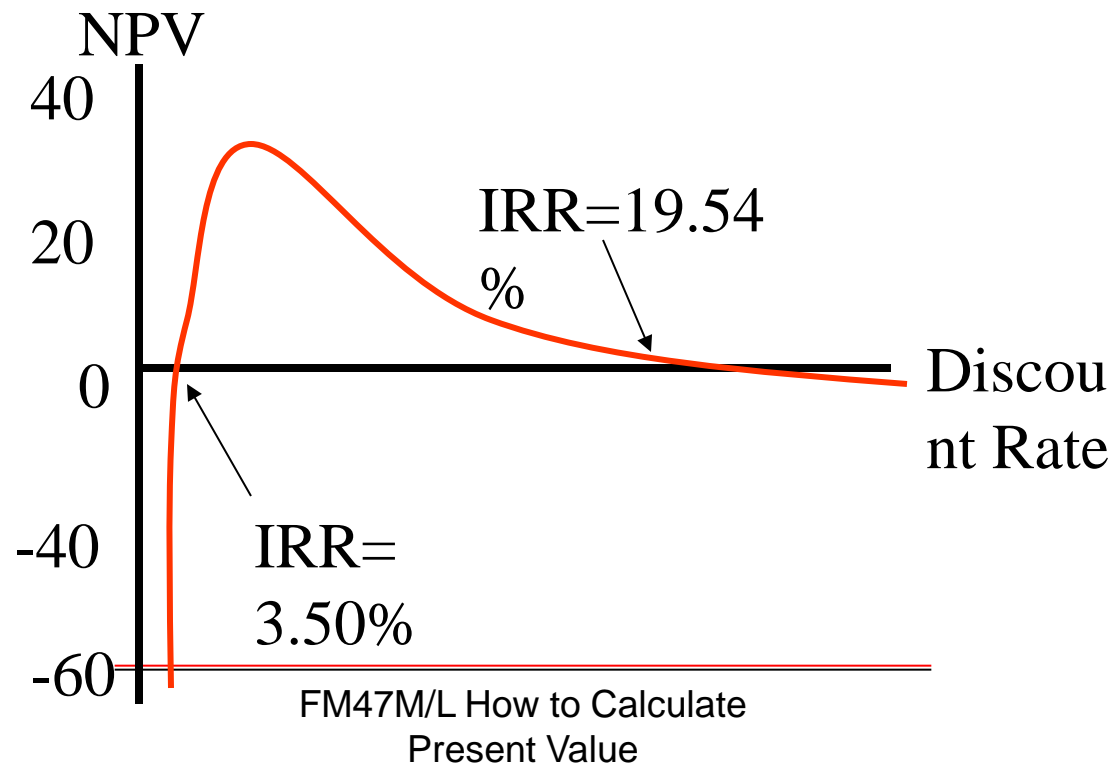
---

- There may be multiple IRRs
- Consider a project with an initial investment, cash inflows for 9 years and a final cost:

Cash Flows (in millions)				
$C_0$	$C_1$	...	$C_9$	$C_{10}$
-30	10		10	-65

# Pitfalls of Using IRR

- At 10% opportunity cost of capital, the NPV is 2,53 million
- The IRRs are +3.50% and +19.54%.



# Simple and Compound Interest

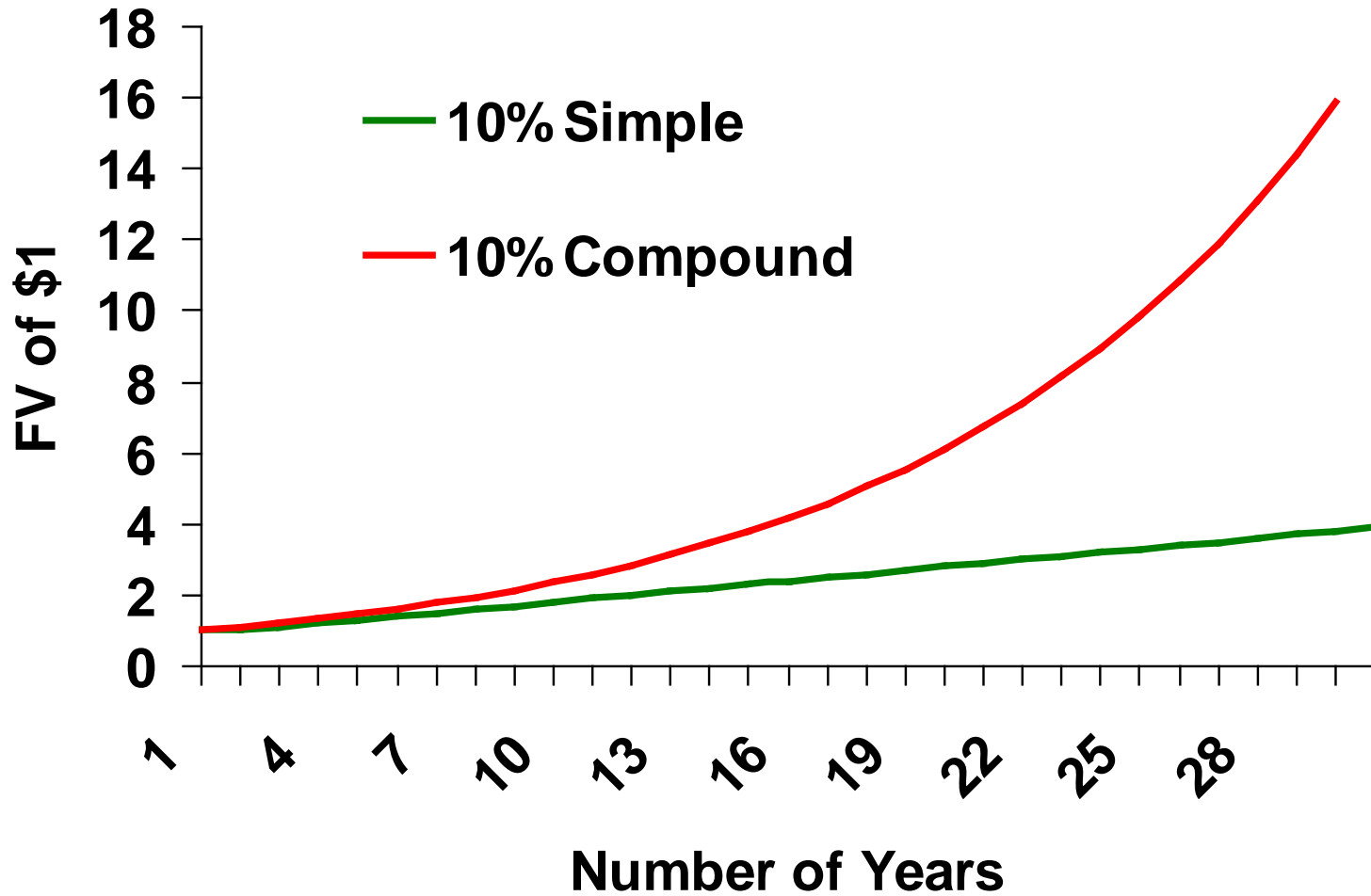
---

- Simple interest means interest is paid (in each period) only on the initial investment
- Compound interest means each interest payment is reinvested to earn more interest in subsequent periods.
- Difference in wealth invested at simple and compounded interest grows with number of periods



# Compound Interest

---



# Quoted vs Effective Interest Rate

---

- In US, companies/banks quote annual percentage rate (APR), an annual rate  $r$  to be paid in  $m$  sub-periods at a  $m$ -period rate of  $r/m$
- For instance, an APR of 12% with interest monthly paid, means a monthly rate of 1%
- The true annual rate is:

$$(1+0.01)^{12} - 1 = 12.68\%$$

- The true rate 12.6% is called the effective annual rate

# Compound Interest

---

i	ii	iii	iv	v
m periods per year	Interest per period r/m	Quoted Rate i × ii	Value after one year	Effective Annual rate
1	6.0000%	0.06	1.06000	6.000%
2	3.0000%	0.06	1.06090	6.090%
4	1.5000%	0.06	1.06136	6.136%
12	0.5000%	0.06	1.06168	6.168%
52	0.1154%	0.06	1.06180	6.180%
365	0.0164%	0.06	1.06183	6.183%

---

General formula for m compounding periods per year:

$$\text{EAR} = \left( 1 + \frac{\text{APR}}{m} \right)^m - 1$$

# Compounded Interest

---

- An investment of £1 at an annual rate  $r$  compounded  $m$  times grows by the end of the year to:  $[1+(r/m)]^m$ .
- Hence, the annually compounded rate of interest is:  
 $[1+(r/m)]^m - 1$
- When  $m$  approaches infinity, interest gets continuously compounded and  $[1+(r/m)]^m$  approaches  $e^r$ , where  $e=2.718281828\dots$ . This is called continuous compounding:

$$\lim_{m \rightarrow \infty} \left(1 + \frac{r}{m}\right)^m = e^r$$

# Equivalent Compounded Rate

---

- Under continuous compounding the present value formula becomes:

$$PV = C_0 + C_1e^{-r_1} + C_2e^{-r_2 \times 2} + \dots + C_Te^{-r_T \times T}$$

- This is because the PV of a single cash flow  $C_t$  is given by:

$$PV(C_t) = \lim_{m \rightarrow \infty} \frac{C_t}{(1 + r_t/m)^{mt}} = \frac{C_t}{e^{r_t \times t}} = C_t e^{-r_t \times t}$$

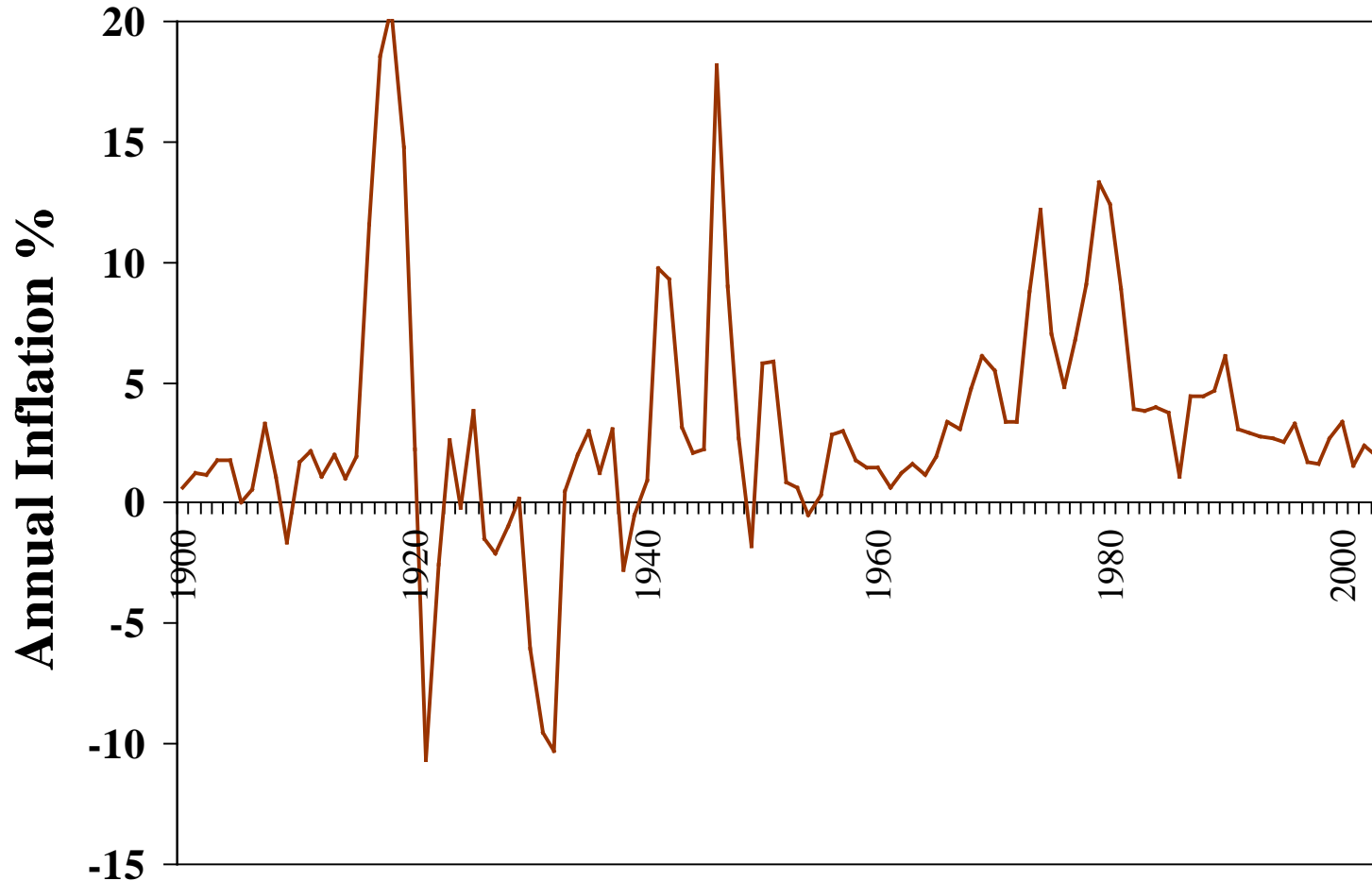
# Nominal and Real Rates

---

- *Inflation*: Rate at which general level of prices increases.
- *Nominal Interest Rate*: Rate at which the value of investment grows in money (£) terms.
- *Real Interest Rate*: Rate at which the purchasing power of an investment increases.

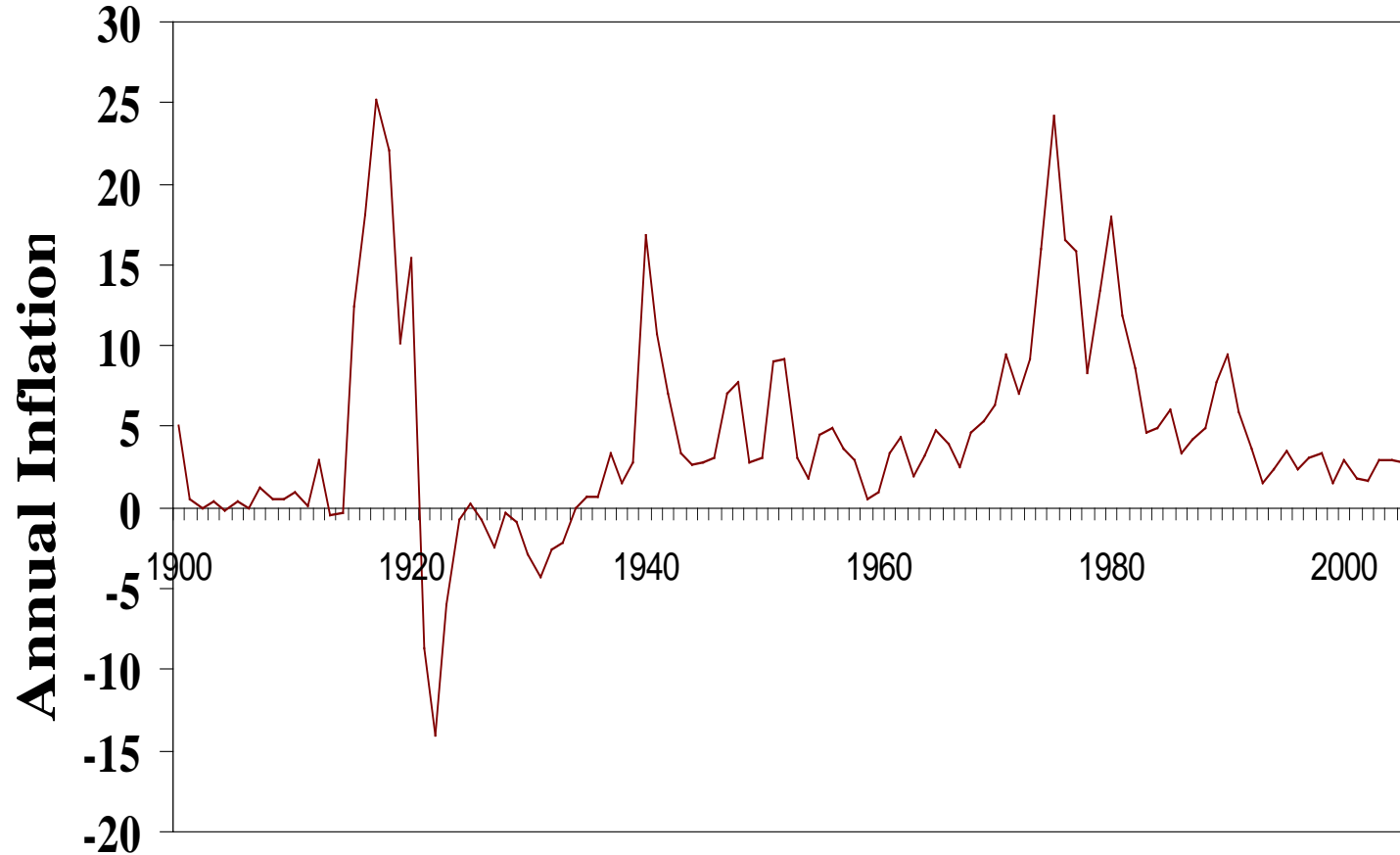
# Inflation

## 100 Years of U.S. Inflation



# Inflation

## 100 Years of UK Inflation





# Real Rate of Interest

---

- Let  $r_n$  be the nominal interest rate,  $r_r$  the real interest rate, and  $i$  the inflation rate:

$$1 + r_r = \frac{1 + r_n}{1 + i}$$

- Rearranging

$$\begin{aligned}(1 + r_r)(1 + i) &= (1 + r_n) \rightarrow 1 + r_r + i + r_r \cdot i = 1 + r_n \\ &\rightarrow r_r + i + r_r \cdot i = r_n\end{aligned}$$

- Given  $r_r \cdot i$  is very small, i.e., 0, we get as approximation

$$r_r + i \approx r_n \text{ OR } r_r \approx r_n - i$$

# Example

---

- If the interest rate on one year government bonds is 5.9% and the inflation rate is 3.3%, what is the real interest rate? What is the approximate real interest rate?

$$1 + r_r = (1+0.059)/(1+0.033) = 1.02517$$

and  $r_r$  is 0.02517 or 2.517%

Approximate:  $r_r = 0.059 - 0.033 = 0.026$  or 2.6%

## Quick Question

---

- You have the opportunity to invest in the Belgravian Republic at 25 percent interest. The inflation rate is 21 percent. What is the real rate of interest? What is the approximate real rate of interest?

# Answer

---

# Summary of Important Equations

---

- Present Value: 
$$PV = C_0 + \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \dots + \frac{C_T}{(1+r)^T}$$
- Perpetuity: 
$$PV = \frac{C}{r}$$
- t-period Annuity: 
$$PV = \frac{C}{r} - \frac{C}{r} \cdot \frac{1}{(1+r)^t}$$
- Growing perpetuity: 
$$PV = \frac{C}{r-g}$$
- t-period growing annuity: 
$$PV = \frac{C_1}{r-g} - \left( \frac{1+g}{1+r} \right)^t \frac{C_1}{r-g}$$
- Real interest rate: 
$$1 + r_r = \frac{1 + r_n}{1 + i} \quad \text{or} \quad r_r \approx r_n - i$$